Mathematics Department

Summer Semester 2020/2021

Instructor: Dr. Ala Talahmeh

Course Code: Math 2311

Title: Calculus III

12

VECTORS AND THE GEOMETRY OF SPACE

12.1 Three-Dimensional Coordinate Systems

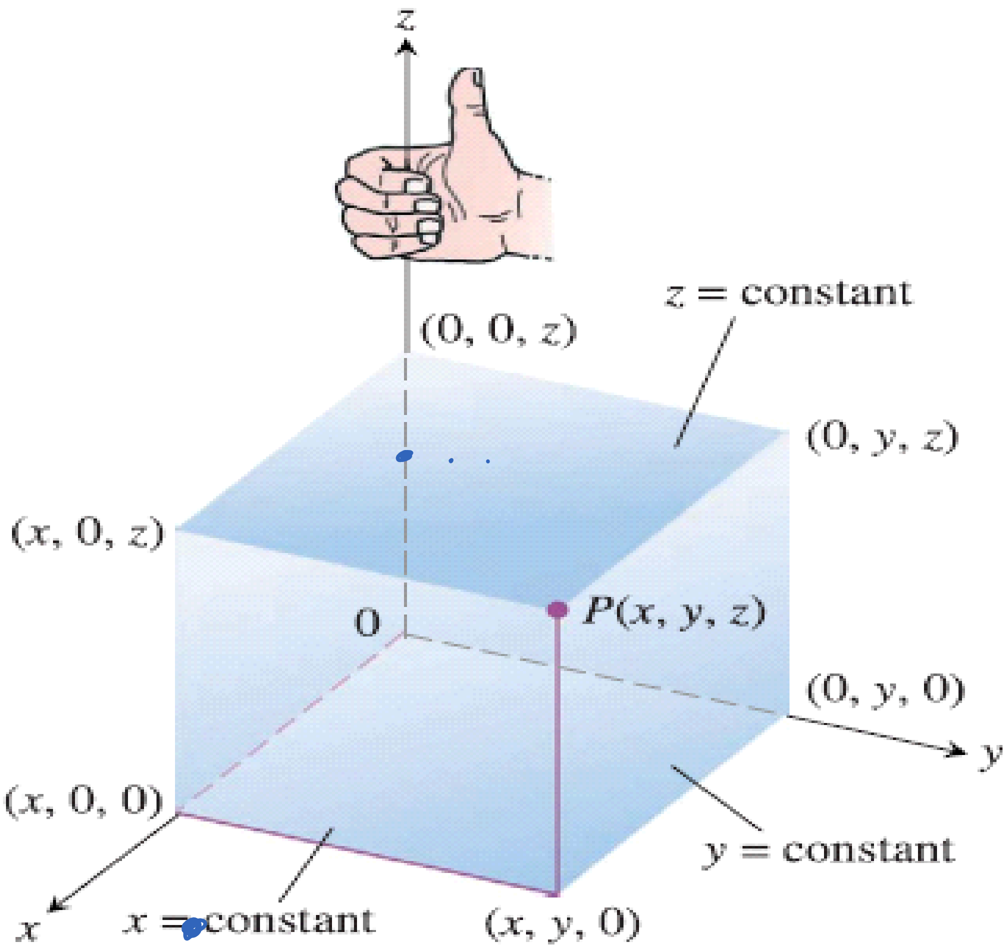
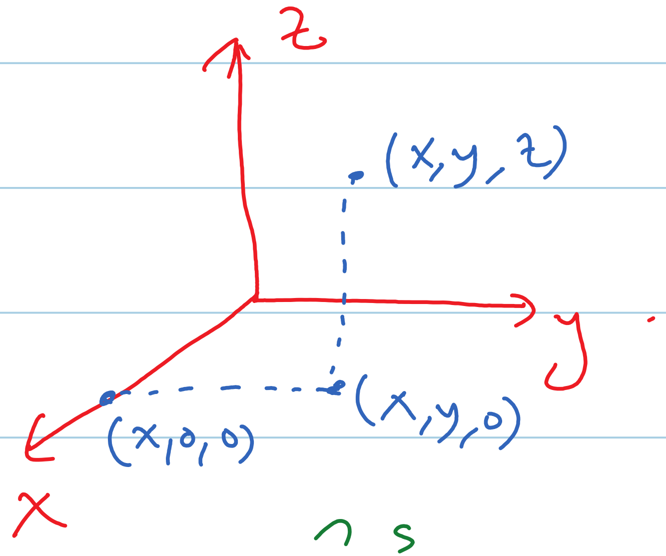


FIGURE 12.1 The Cartesian coordinate system is right-handed.



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8 octants

First octant

$x \geq 0, y \geq 0,$
$z \geq 0.$

xy-plane $(z=0)$

xz-plane $(y=0)$

yz-plane $(x=0)$

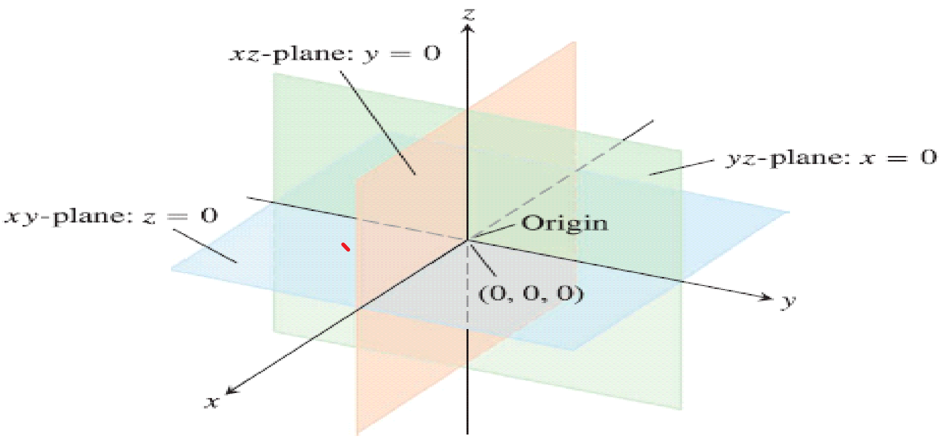
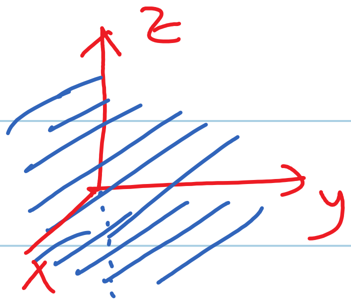


FIGURE 12.2 The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

EXAMPLE 1 We interpret these equations and inequalities geometrically.



(Describe in space).

① $z \geq 0$: the half space consisting of all points above xy-plane.

② $x = -3$: plane perpendicular to x-axis at $x=-3$, parallel to yz-plane

xy-plane

③ $\boxed{z=0}$, $x \leq 0$, $y \geq 0$: second quadrant in xy-plane.

④ $-1 \leq y \leq 1$: the slab between the planes $y=-1$ and $y=1$

⑤ $y=-2$, $z=2$: the line in which the planes $y=-2$ and $z=2$ intersect.

⑥ $x^2+y^2=4$, $z=3$: Circle in the plane $z=3$

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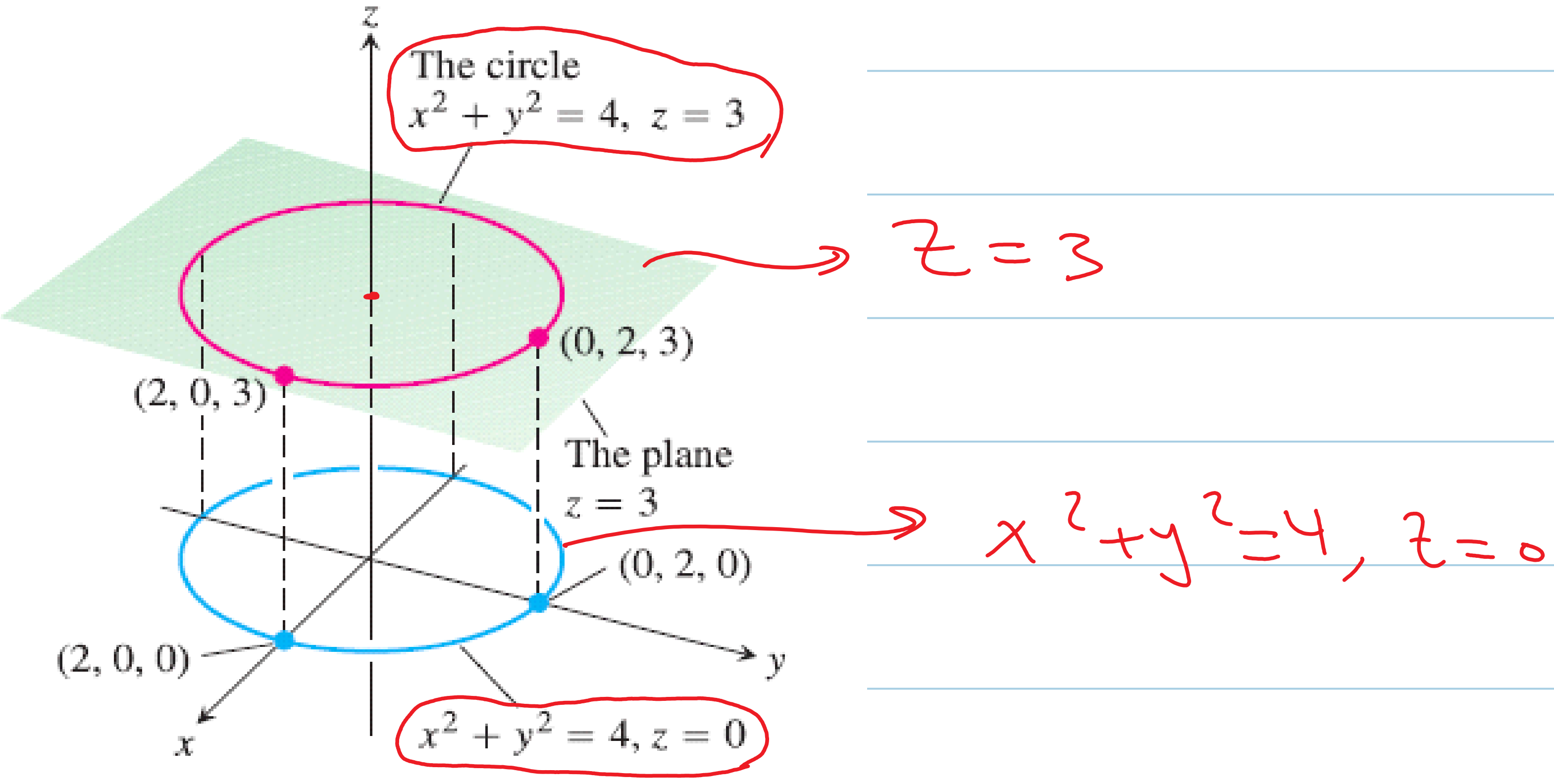


FIGURE 12.4 The circle $x^2 + y^2 = 4$ in the plane $z = 3$ (Example 2).

Distance and Spheres in Space

The formula for the distance between two points in the xy-plane extends to points in space.

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Ex. $P_1(\overset{x_1}{2}, \overset{y_1}{3}, \overset{z_1}{4})$, $P_2(\overset{x_2}{-1}, \overset{y_2}{5}, \overset{z_2}{0})$, find $|P_1P_2|$

Sol. $$|P_1P_2| = \sqrt{(-1-2)^2+(5-3)^2+(0-4)^2}$$
$$= \sqrt{9+4+16} = \sqrt{29}.$$

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

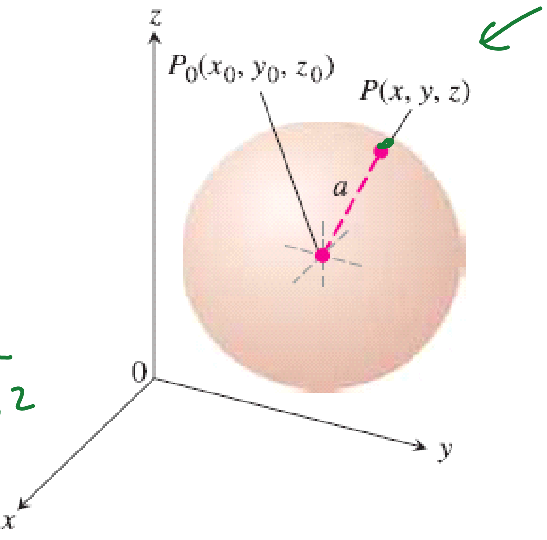


FIGURE 12.6 The sphere of radius a centered at the point (x_0, y_0, z_0).

$$a = \text{radius} = |P_0P| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

$$\Rightarrow \boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2}$$

EXAMPLE 4 Find the center and radius of the sphere

$$\boxed{x^2} + y^2 + z^2 + \boxed{3x} - 4z + 1 = 0.$$

Sol.

$$\underbrace{x^2+3x+\left(\frac{3}{2}\right)^2}+y^2+z^2-4z+\left(\frac{-4}{2}\right)^2 = -1+\left(\frac{3}{2}\right)^2+\left(\frac{-4}{2}\right)^2$$

$$\left(x+\frac{3}{2}\right)^2+(y-0)^2+(z-2)^2 = -1+\frac{9}{4}+4$$

$$\left(x+\frac{3}{2}\right)^2+(y-0)^2+(z-2)^2 = \frac{21}{4}$$

radius $= a = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$

Center $\left(-\frac{3}{2}, 0, 2\right)$.

EXAMPLE 5 Here are some geometric interpretations of inequalities and equations involving spheres.

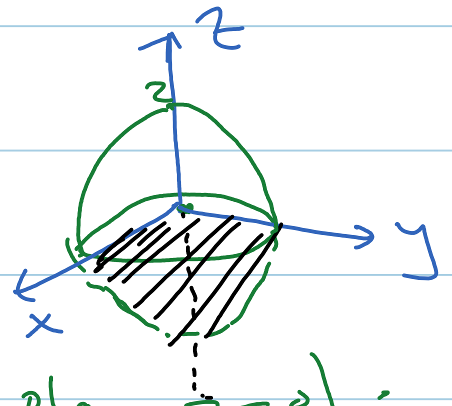
(a) $x^2 + y^2 + z^2 < 4$ The interior points of the sphere $x^2+y^2+z^2=4$

(b) $x^2 + y^2 + z^2 \leq 4$ the sphere $x^2+y^2+z^2=4$ together with its interior.
(the solid ball bounded by the sphere $x^2+y^2+z^2=4$).

(c) $x^2 + y^2 + z^2 > 4$ the exterior of the sphere $x^2+y^2+z^2=4$.

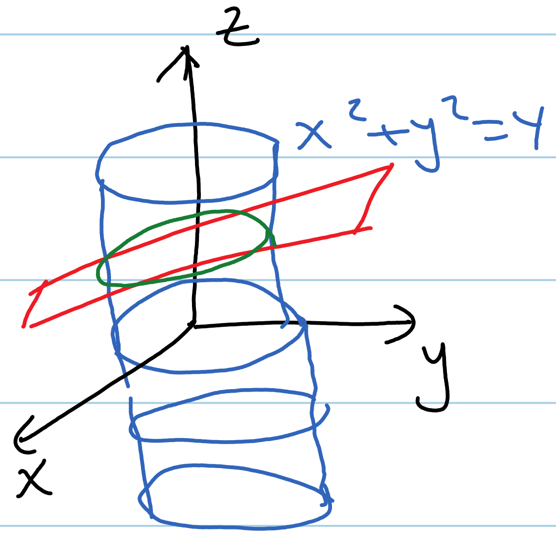
(d) $x^2 + y^2 + z^2 = 4, z \leq 0$

The lower hemisphere cut from the sphere $x^2+y^2+z^2=4$ by the xy-plane (the plane $z=0$).



In Exercises 1–16, give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations.

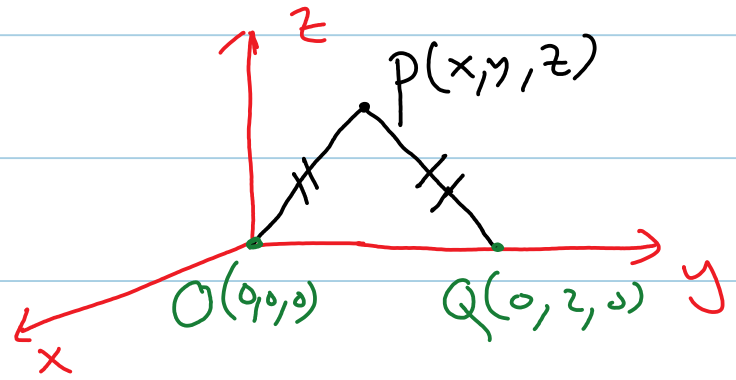
13. $\underbrace{x^2 + y^2 = 4}_{\text{cylinder}}$, $\boxed{z = y}$



the ellipse formed by the intersection of the Cylinder $x^2+y^2=4$ and the plane $z=y$.

In Exercises 25–34, describe the given set with a single equation or with a pair of equations.

32. The set of points in space equidistant from the origin and the point (0, 2, 0)



$$|\overline{PO}| = |\overline{PQ}|$$

$$\sqrt{x^2+y^2+z^2} = \sqrt{x^2+(y-2)^2+z^2}$$

$$\Rightarrow y^2 = (y-2)^2$$

$$y^2 = y^2-4y+4 \Rightarrow \boxed{y=1}$$

34. The set of points in space that lie 2 units from the point (0, 0, 1) and, at the same time, 2 units from the point (0, 0, −1)

$P(0,0,1)$, $Q(0,0,-1)$ $R(x,y,z)$.

$|\overline{PR}| = 2$, $|\overline{QR}| = 2$

$\sqrt{x^2+y^2+(z-1)^2} = 2$, $\sqrt{x^2+y^2+(z+1)^2} = 2$

$x^2+y^2+(z-1)^2=4$ ---- ①

$x^2+y^2+(z+1)^2=4$ ---- ②

① & ② $\Rightarrow$

$\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-2z+\cancel{1}=\cancel{x^2}+\cancel{y^2}+\cancel{z^2}+2z+\cancel{1}$

$\Rightarrow 4z=0 \Rightarrow \boxed{z=0}$ ... ③

③ into ① $\Rightarrow x^2+y^2=3$

$\therefore$ $x^2+y^2=3$, $z=0$.

(the circle $x^2+y^2=3$ in the xy-plane).

⑧ $y^2+z^2=1$, $x=0$.

The Circle $y^2+z^2=1$ in the yz-plane.

⑫ $x^2+(y-1)^2+z^2=4$, $y=0$.

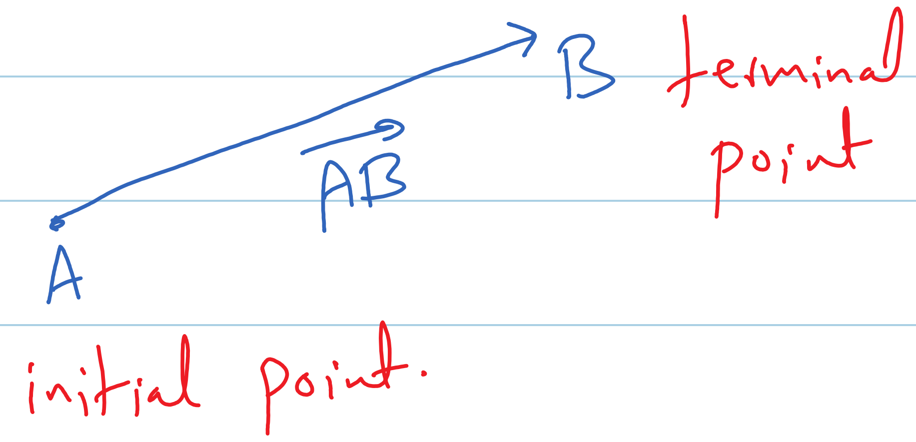
Sol. $x^2+(-1)^2+z^2=4 \Rightarrow \boxed{x^2+z^2=3}$

the Circle $x^2+z^2=3$ in the xz-plane.

⑭ $x^2+y^2+z^2=4$, $y=x$.

The circle formed by the intersection of the sphere $x^2+y^2+z^2=4$ and the plane $y=x$.

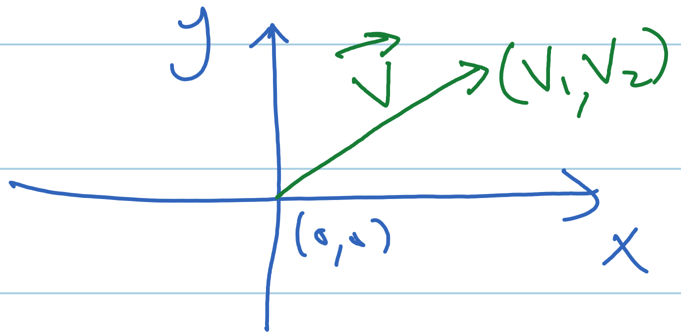
12.2 Vectors



· length of $\overrightarrow{AB}$ or the magnitude of $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$.

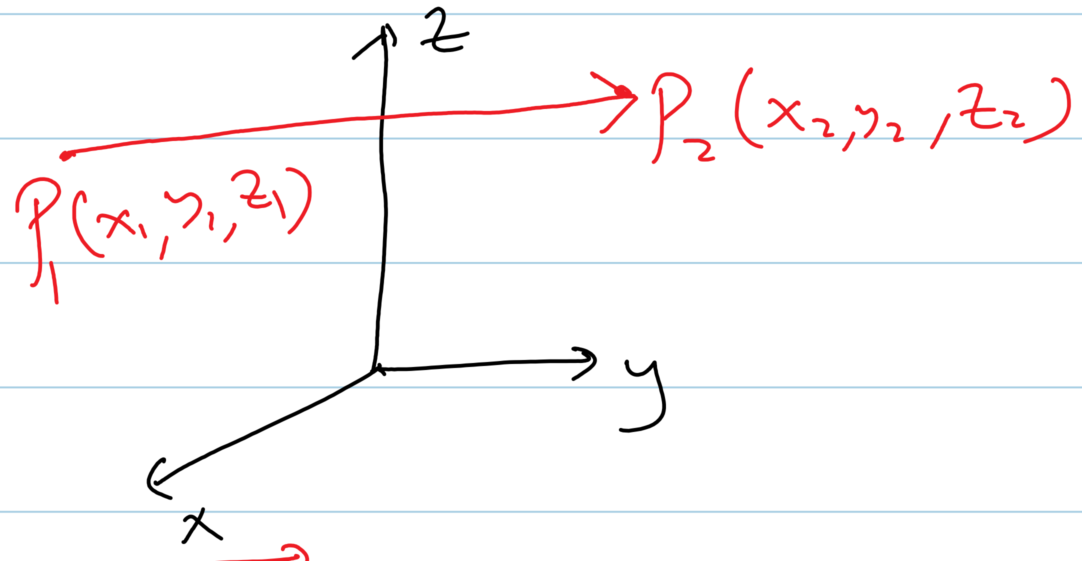
· Two vectors are equal if they have the same length and direction.

Two dimension (standard)



$\vec{V} = \langle V_1, V_2 \rangle$

Three dimension $\vec{V} = \langle V_1, V_2, V_3 \rangle$.



$\overrightarrow{P_1P_2} = \langle x_2-x_1, y_2-y_1, z_2-z_1 \rangle$
(direction)

magnitude $= |\overrightarrow{P_1P_2}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Ex. $P(-3,4,1)$, $Q(-5,2,2)$.

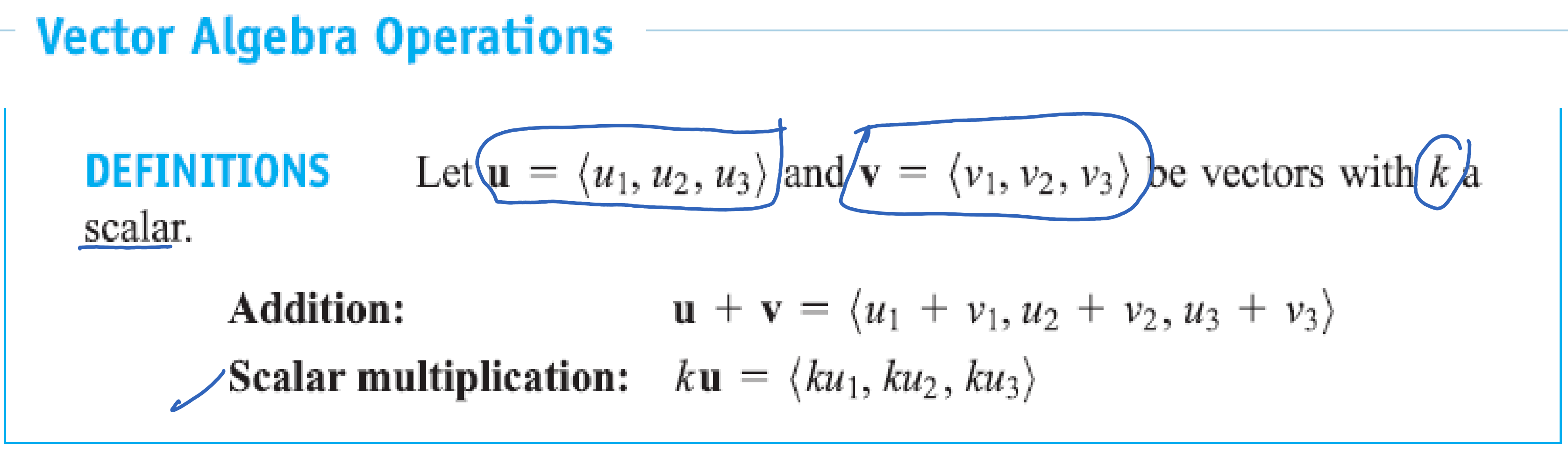
Find the components & direction of $\vec{PQ}$.

Sol. $\vec{PQ} = \langle -5+3,\ 2-4,\ 2-1 \rangle$

$= \langle -2, -2, 1 \rangle$

length $= |\vec{PQ}| = \sqrt{(-2)^2+(-2)^2+(1)^2}$

$= \sqrt{9} = 3$.



Vector Algebra Operations

DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

Ex. $\vec{u} = \langle -1, 3, 1 \rangle$, $\vec{v} = \langle 4, 7, 0 \rangle$

(a) $\vec{u} + 3\vec{v} = \langle -1,3,1 \rangle + 3\langle 4,7,0 \rangle$

$= \langle -1,3,1 \rangle + \langle 12,21,0 \rangle$

$= \langle -1+12,\ 3+21,\ 1+0 \rangle$

$= \langle 11, 24, 1 \rangle$

(b) $\left| \frac{1}{2}\vec{v} \right| = \left| \frac{1}{2}\langle 4,7,0 \rangle \right|$

$= \left| \langle 2, \frac{7}{2}, 0 \rangle \right|$

$= \sqrt{2^2 + \left(\frac{7}{2}\right)^2 + (0)^2} = \frac{\sqrt{65}}{2}$

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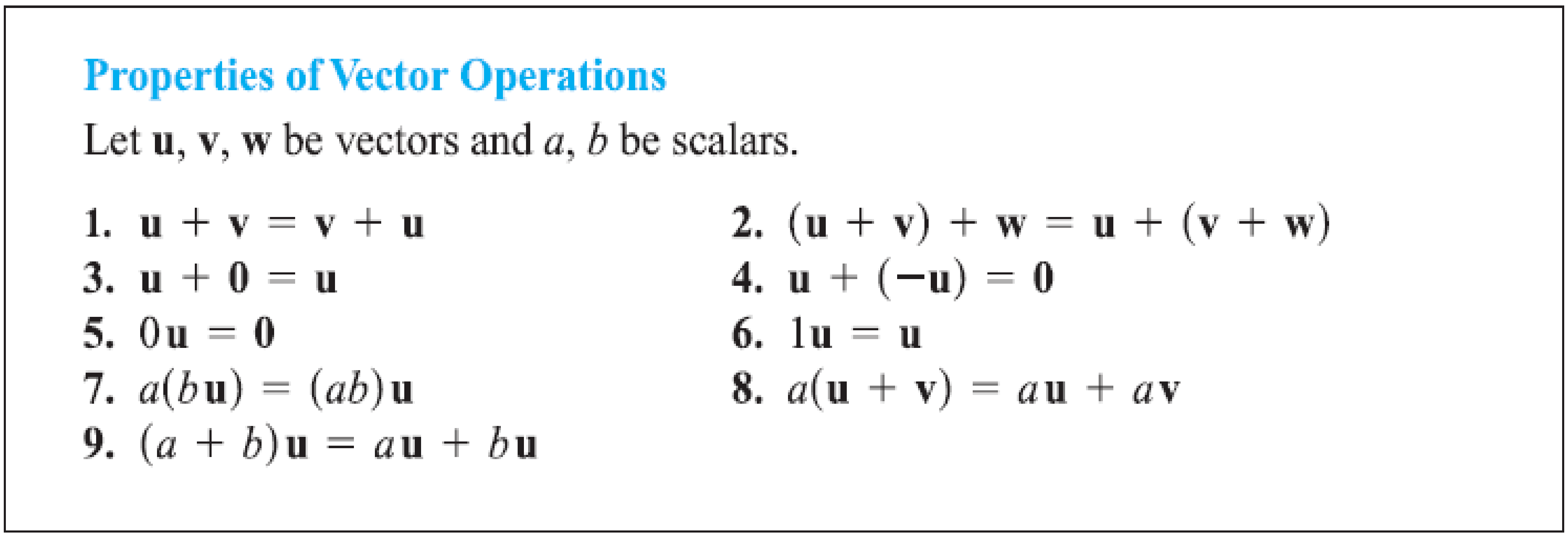


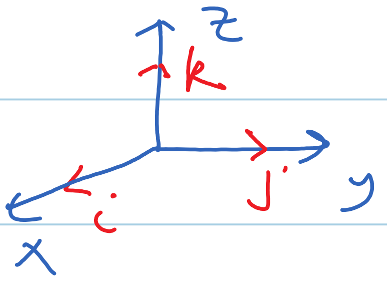
Properties of Vector Operations

Let **u**, **v**, **w** be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit vectors

A vector of length 1 is called unit vector.



Standard unit vectors

$i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$

$$\boxed{\vec{V} = \langle x, y, z \rangle}$$

$$= \langle x, 0, 0 \rangle + \langle 0, y, 0 \rangle + \langle 0, 0, z \rangle$$

$$= x\langle 1, 0, 0 \rangle + y\langle 0, 1, 0 \rangle + z\langle 0, 0, 1 \rangle$$

$$= xi + yj + zk$$ ✓

ex. $\vec{V} = \langle 2, 3, 5 \rangle = 2i + 3j + 5k$.

Rmk. If $\vec{u} \neq \vec{0}$, then $\frac{\vec{u}}{|\vec{u}|}$ is a unit vector in the direction of $\vec{u}$.

$-\frac{\vec{u}}{|\vec{u}|}$ is a unit vector in the opposite direction of $\vec{u}$.

Ex. Find a unit vector $\vec{v}$ in the direction of the vector $\overrightarrow{P_1P_2}$ where $P_1(1,0,1)$, $P_2(3,2,0)$.

Sol.

$$\vec{v} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{(3-1)i+(2-0)j+(0-1)k}{\sqrt{4+4+1}}$$

$$= \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k.$$

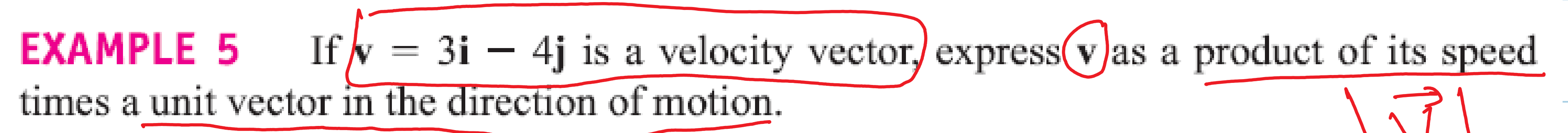
Q33) Find a vector $\vec{w}$ of length 7 in the direction of $\vec{v} = 12i - 5k$.

Solution.

$$\vec{w} = 7\frac{\vec{v}}{|\vec{v}|} = \frac{7}{\sqrt{144+25}}(12i-5k).$$

$$= \frac{7}{13}(12i-5k).$$

$$= \frac{84}{13}i - \frac{35}{13}k.$$



EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times a unit vector in the direction of motion.

Sol.

$$\vec{v} = \underbrace{|\vec{v}|}_{\text{Speed}} \overbrace{\left(\frac{\vec{v}}{|\vec{v}|}\right)}^{\text{unit vector in the direction of motion}}, \quad |\vec{v}| = \sqrt{3^2+(-4)^2} = 5$$

$$= 5\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

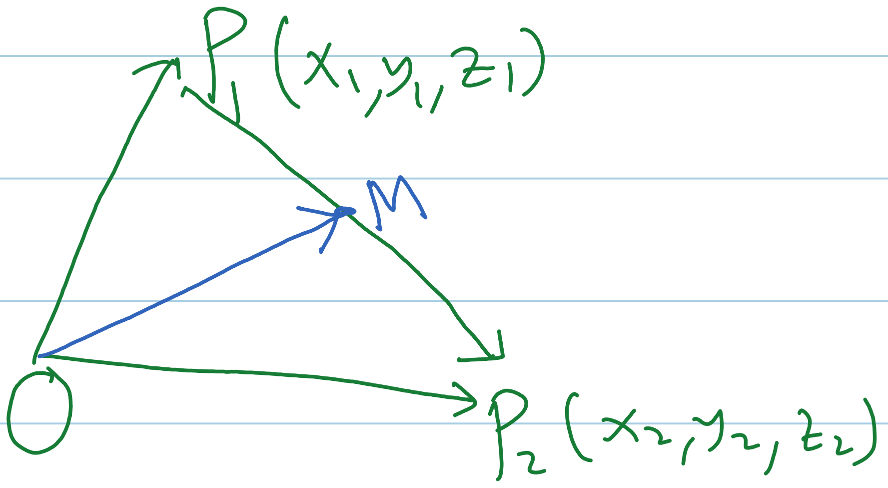
Summary. If $\vec{v} \neq \vec{0}$, then

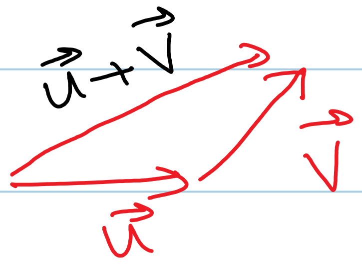
① $\frac{\vec{v}}{|\vec{v}|}$ is aunit vector in the direction of $\vec{v}$.

② The equation $\vec{v} = |\vec{v}| \frac{\vec{v}}{|\vec{v}|}$ expresses as its length times its direction.

Midpoint of a line segment

The Midpoint M of a line segment joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$





$$\overrightarrow{OM} = \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2}$$

$$= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1})$$

$$= \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2})$$

$$= \frac{1}{2}(x_1 i + y_1 j + z_1 k + x_2 i + y_2 j + z_2 k)$$

$$\vec{OM} = \frac{x_1+x_2}{2} i + \frac{y_1+y_2}{2} \hat{j} + \frac{z_1+z_2}{2} k$$

$$\therefore M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right).$$

Ex: the midpoint of the segment joining $P_1(3,-2,0)$, $P_2(7,4,0)$ is

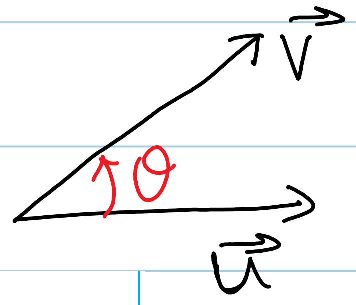
$$M\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+0}{2} \right) = (5,1,0).$$

12.3 The Dot Product

DEFINITION The **dot product** $\mathbf{u} \cdot \mathbf{v}$ ("**u** dot **v**") of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

OR $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$



THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}||\mathbf{v}|}\right).$$

$, 0 \leq \theta < \pi$

Rmk. ① Two vectors are orthogonal (perpendicular / normal) iff $\vec{u} \cdot \vec{v} = 0$

② $\vec{0}$ is orthogonal to any vector $\vec{u}$.

Since $\vec{0} \cdot \vec{u} = 0 \cdot u_1 + 0u_2 + 0u_3 = 0$

Ex. $\vec{u} = 3i - 2j + k$ and $\vec{v} = 2j + 4k$ are orthogonal since

$\vec{u} \cdot \vec{v} = (3)(0) + (-2)(2) + (1)(4)$

$= -4 + 4 = 0$.

Ex. If $\vec{u} = 3i - 2j + k$ and $\vec{v} = 2j + xk$ are orthogonal, find x.

Sol. $\vec{u} \cdot \vec{v} = 0 \Rightarrow (3)(0) + (-2)(2) + (1)(x) = 0$

$-4 + x = 0 \Rightarrow \boxed{x = 4}$

Ex. Find the angle between the vectors $\vec{u} = i-2j-k$ and $\vec{v} = 6i+3j+2k$.

Solution

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$$= \cos^{-1}\left(\frac{(1)(6)+(-2)(3)+(-1)(2)}{\sqrt{1+4+1}\,\sqrt{36+9+4}}\right)$$

$$= \cos^{-1}\left(\frac{-2}{7\sqrt{6}}\right) \approx \ldots\ldots$$

Properties of the Dot Product

If **u**, **v**, and **w** are any vectors and c is a scalar, then

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
2. $(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$
3. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
4. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0}\cdot\mathbf{u} = 0.$

Proof. ④ $\vec{u}\cdot\vec{u} = u_1u_1 + u_2u_2 + u_3u_3$

$$= \left(\sqrt{u_1^2+u_2^2+u_3^2}\right)^2$$

$$= |\vec{u}|^2$$

Ex. If $|\vec{u}|=4$, $|\vec{v}|=5$, $\theta = \pi/3$ is the angle between $\vec{u}$ and $\vec{v}$. Find $|\vec{u}+\vec{v}|$.

Sol. $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\frac{\pi}{3} = (4)(5)(\frac{1}{2}) = 10.$

Using ④

$$|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

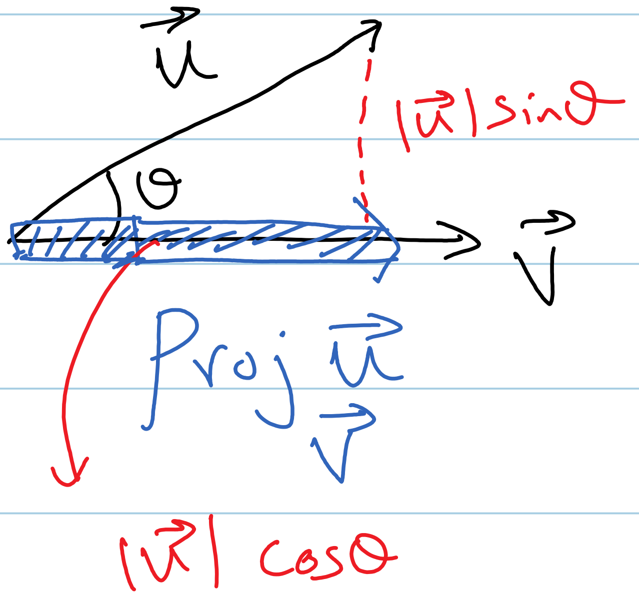
$$= |\vec{u}|^2 + 2(\vec{u}\cdot\vec{v}) + |\vec{v}|^2$$

$$= (4)^2 + 2(10) + (5)^2$$

$$|\vec{u}+\vec{v}|^2 = 16 + 20 + 25 = 61$$

$$\therefore\ |\vec{u}+\vec{v}| = \sqrt{61}\,.$$

Vector Projection



$$\text{Proj}_{\vec{v}}\vec{u} = (\text{length})(\text{direction})$$

$$= (|\vec{u}|\cos\theta)\,\frac{\vec{v}}{|\vec{v}|}$$

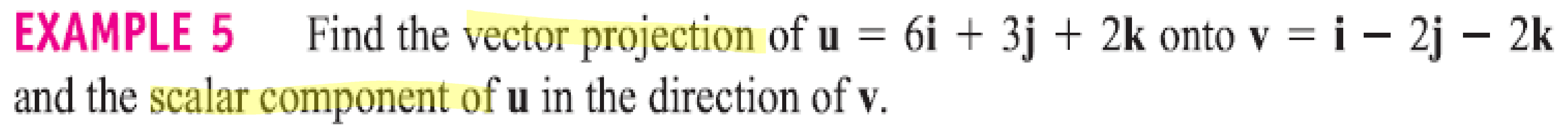
$$= \left(\cancel{|\vec{u}|}\,\frac{\vec{u}\cdot\vec{v}}{\cancel{|\vec{u}|}\,|\vec{v}|}\right)\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

$$\boxed{\text{Proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}}$$

is the vector projection of $\vec{u}$ onto $\vec{v}$.

. Scalar component of $\vec{u}$ onto $\vec{v}$ is

$$\boxed{\text{Comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}$$



EXAMPLE 5 Find the vector projection of **u** = 6**i** + 3**j** + 2**k** onto **v** = **i** − 2**j** − 2**k** and the scalar component of **u** in the direction of **v**.

Sol:

$$\text{Proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$$

$$= \left(\frac{6-6-4}{\left(\sqrt{1+4+4}\right)^2} \right) (i - 2j - 2k)$$

$$= \frac{-4}{9} (i - 2j - 2k)$$

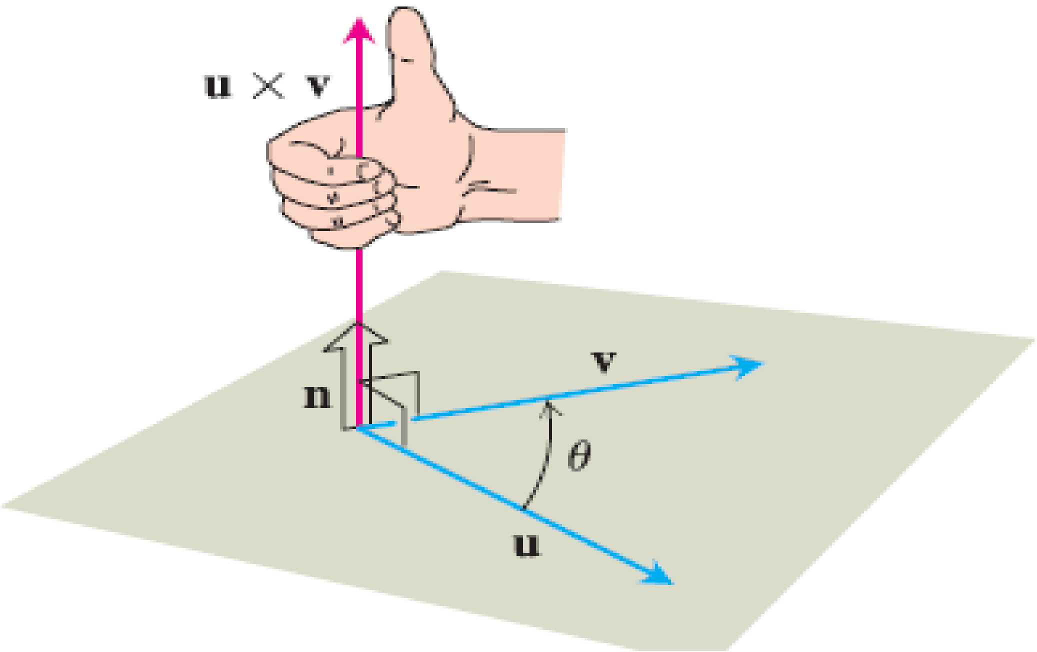
$$= \frac{-4}{9} i + \frac{8}{9} j + \frac{8}{9} k .$$

$$\text{Comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{6-6-4}{\sqrt{9}} = -\frac{4}{3} .$$

12.4 The Cross Product

$$\vec{u} \times \vec{v} = \left(|\vec{u}|\,|\vec{v}| \sin\theta \right) \vec{n}$$

$\vec{n}$: is the unit vector normal to the plane containing $\vec{u}$ and $\vec{v}$



Rmk. Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel iff $\vec{u} \times \vec{v} = \vec{0}$

Properties of the Cross Product

If **u**, **v**, and **w** are any vectors and *r*, *s* are scalars, then

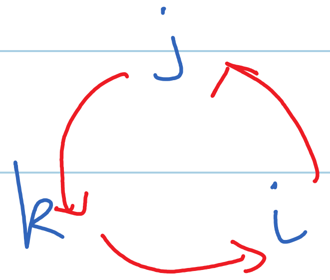
1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

$$i\times j = k$$

$$j\times k = i$$

$$k\times i = j$$

$$i\times i = \vec{0} = j\times j = k\times k$$



Determinant Formula for u × v

$$\vec{u} = u_1 i + u_2 j + u_3 k \ , \ \vec{v} = v_1 i + v_2 j + v_3 k$$

$$\vec{u}\times\vec{v} = \langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle$$

$$= (\underbrace{u_1 i}\, + u_2 j + u_3 k)\times(\underbrace{v_1 i + v_2 j + v_3 k})$$

$$= \cancelto{\vec{0}}{(u_1v_1)\ i\times i} + (u_1v_2)\overset{k}{\boxed{i\times j}} + (u_1v_3)\overset{-j}{\boxed{i\times k}}$$

$$+ (u_2v_1)\overset{-k}{\boxed{j\times i}} + \cancelto{\vec{0}}{(u_2v_2)\ j\times j} + (u_2v_3)\overset{i}{\boxed{j\times k}}$$

$$+ (u_3v_1)\underset{j}{\boxed{k\times i}} + (u_3v_2)\underset{-i}{\boxed{k\times j}} + \cancelto{\vec{0}}{(u_3v_3)\,k\times k}$$

$$= (u_2v_3 - u_3v_2)\,i + (u_3v_1 - u_1v_3)\,j + (u_1v_2 - u_2v_1)\,k$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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Ex. $\vec{u} = 2i + j + k$, $\vec{v} = -4i + 3j + k$

a) $\vec{u} \times \vec{v} = \begin{vmatrix} \overset{\oplus}{i} & \overset{\ominus}{j} & \overset{\oplus}{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$

$= (1-3)i - (2 - -4)j + (6 - -4)k$

$= -2i - 6j + 10k.$

b) Find a unit vector normal to the plane containing $\vec{u}$ and $\vec{v}$.

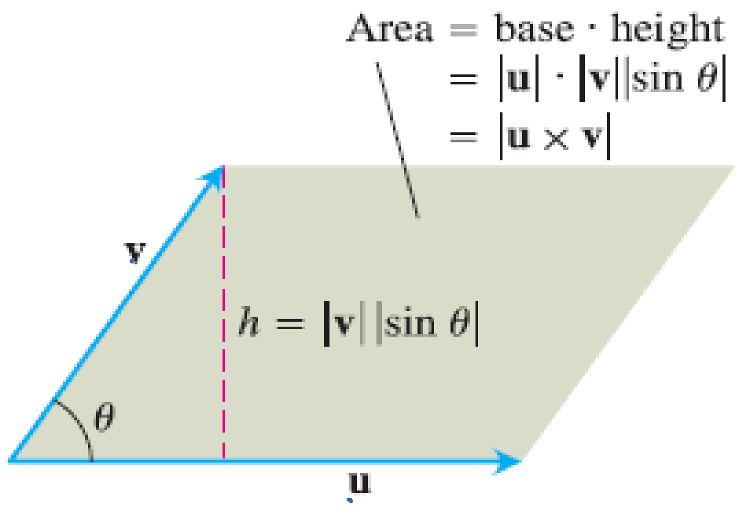
Sol. $\dfrac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \dfrac{1}{\sqrt{4+36+100}}(-2i - 6j + 10k)$

$= \dfrac{1}{\sqrt{140}}(-2i - 6j + 10k)$

$|\mathbf{u} \times \mathbf{v}|$ Is the Area of a Parallelogram

Because $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin\theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$



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EXAMPLE 2 (a) Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

(b) Find the area of the triangle ΔPQR.

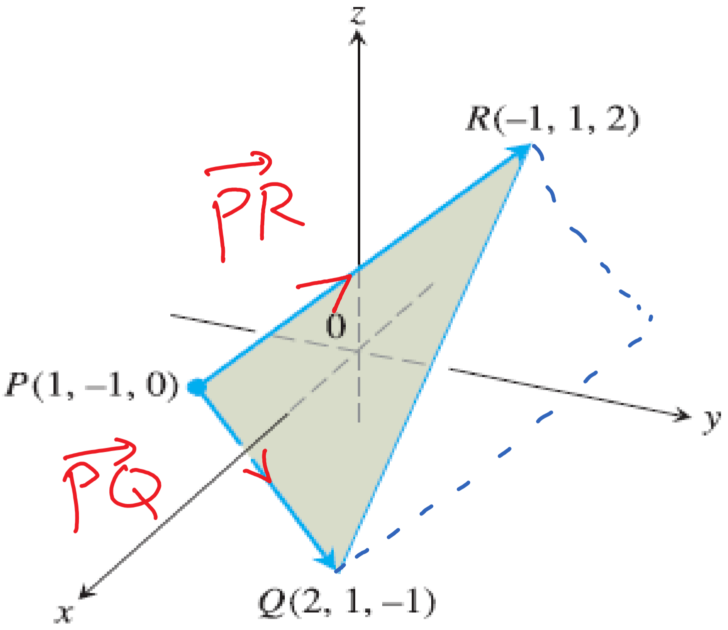
(a) $\overrightarrow{PR} \times \overrightarrow{PQ}$

$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k$

$= -2i + 2j + 2k$

$\overrightarrow{PQ} = (2-1)i + (1--1)j - k$

$= i + 2j - k$



$$\overrightarrow{PR} \times \overrightarrow{PQ} = \begin{vmatrix} i & j & k \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} i - \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} j + \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} k$$

$$= -6i - 0j - 6k = -6i - 6k.$$

(b) Area of the triangle $= \frac{1}{2} \left| \overrightarrow{PR} \times \overrightarrow{PQ} \right|$

$$= \frac{1}{2} \sqrt{(-6)^2 + (-6)^2}$$

$$= \frac{1}{2} \sqrt{72}$$

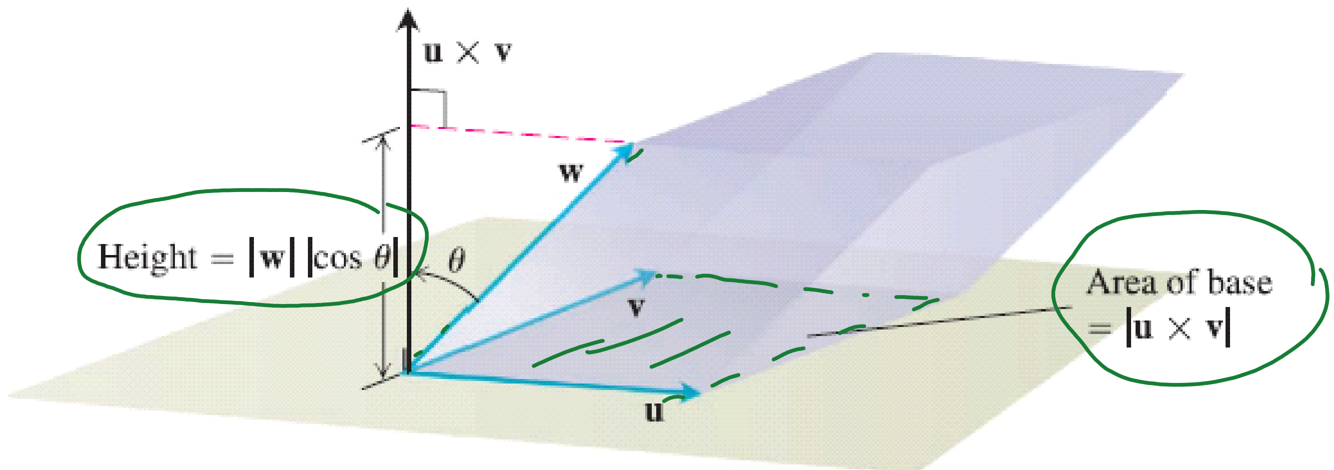
$$= \frac{1}{2} \cdot 6\sqrt{2} = 3\sqrt{2}.$$

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of **u**, **v**, and **w** (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}||\mathbf{w}||\cos\theta|,$$

$$|x \cdot y| = |x| |y| |\cos\theta|$$



Volume = area of base · height
= |u × v| |w| |cos θ|
= |(u × v) · w|

FIGURE 12.34 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} \end{array}$$

$$= 1 \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} - 1 \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}$$

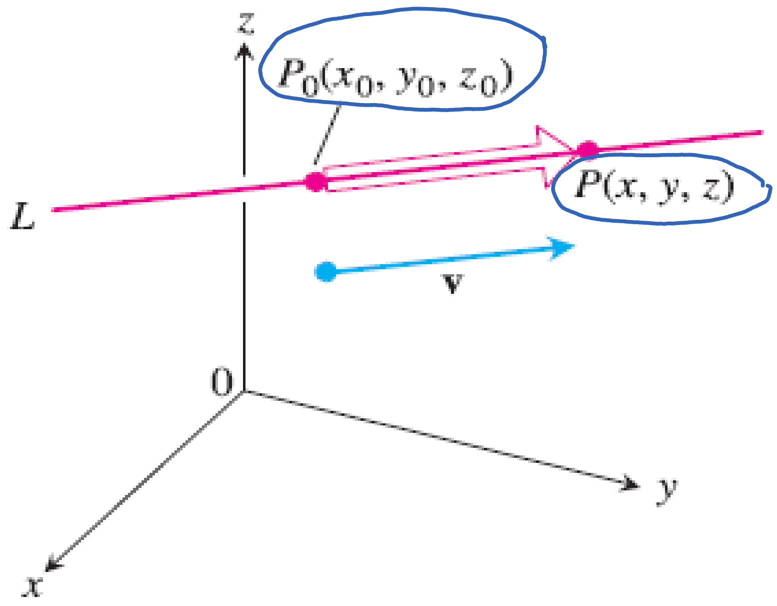
$$= 1(0 - 21) - 2(8 - 0) - 1(-14 - 0)$$

$$= -21 - 16 + 14 = -23.$$

$\therefore$ Volume $= |-23| = 23$.

12.5 Lines and Planes in Space

Lines and Line Segments in Space



$\overrightarrow{P_0P} \parallel \vec{v}$

$\overrightarrow{P_0P} = (x-x_0)i+(y-y_0)j+(z-z_0)k$

$\vec{v} = v_1 i+v_2 j+v_3 k.$

$$L = \{ P(x,y,z) : \overrightarrow{P_0P} \parallel \vec{v} \}$$

Thus, $\overrightarrow{P_0P} = t\vec{v}$, t scalar parameter $-\infty < t < \infty$

$$(x-x_0)i+(y-y_0)j+(z-z_0)k = t(v_1 i+v_2 j+v_3 k)$$

$$\underbrace{\tilde{x}i+yj+zk}_{} = \underbrace{\tilde{x_0}i+y_0j+z_0k}_{}+\tilde{t}(v_1i+v_2j+v_3k)$$

$$\boxed{\vec{r}(t) = \vec{r_0}(t)+t\vec{v}, \quad -\infty<t<\infty}$$

is called vector eq. for the line L through $P_0(x_0,y_0,z_0)$ parallel to $\vec{v}$.

$\vec{r}$: position vector of $P(x,y,z)$ on L.

r_0 : 〃 〃 〃 $P_0(x_0,y_0,z_0)$.

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The standard parametric eqs of the line L through $P_0(x_0,y_0,z_0)$ parallel to $\vec{v} = v_1 i + v_2 j + v_3 k$

are $\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}$, $-\infty < t < \infty$

EXAMPLE 1 Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (Figure 12.36).

Sol.

$$\begin{aligned} x &= -2+2t \\ y &= 0+4t \\ z &= 4-2t \end{aligned} \quad , \quad -\infty < t < \infty.$$

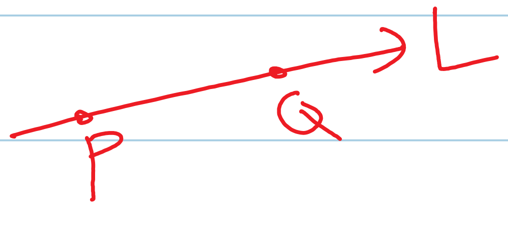
OR $\dfrac{x+2}{2} = \dfrac{y-0}{4} = \dfrac{z-4}{-2}$ are called symmetric eqs.

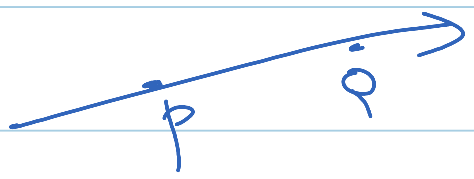
Vector eq. $\underbrace{(x,y,z)}_{r(t)} = \underbrace{(-2,0,4)}_{r_0(t)} + t\vec{v}$

EXAMPLE 2 Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

$\vec{v} = \vec{PQ} = (-3-1)i + (2+1)j + (-3-4)k$

$= -4i + 3j - 7k$



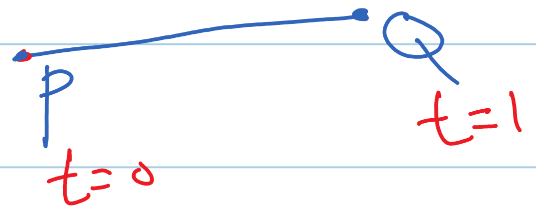


$P(-3,2,-3)$, $\vec{v} = 4i - 3j + 7k$

the Parametric eqs are

$x = -3 + 4t$, $y = 2 - 3t$, $z = -3 + 7t$, $-\infty < t < \infty$

EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure 12.37).



$\vec{v} = \overrightarrow{PQ} = 4i - 3j + 7k$

P_0 is $P(-3, 2, -3)$.

$x = -3 + 4t$, $y = 2 - 3t$, $z = -3 + 7t$, $0 \le t \le 1$

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$P(x_0, y_0, z_0)$, $Q(x_1, y_1, z_1)$

$x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$, $z = z_0 + (z_1 - z_0)t$

$0 \le t \le 1$

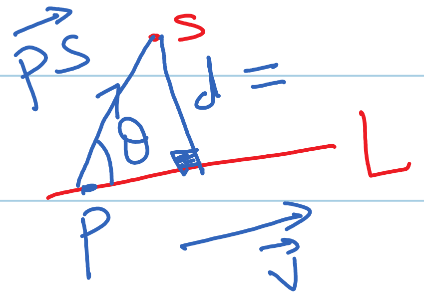
Remark.

The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector $\mathbf{v}$. Rewriting Equation (2), we have

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v}\\ &= \underbrace{\mathbf{r}_0}_{\text{Initial position}} + \underbrace{t}_{\text{Time}}\underbrace{|\mathbf{v}|}_{\text{Speed}}\underbrace{\frac{\mathbf{v}}{|\mathbf{v}|}}_{\text{Direction}}.\end{aligned}$$

$r(t) = r_0 + t\vec{v}$

In other words, the position of the particle at time t is its initial position plus its distance moved (speed × time) in the direction $\mathbf{v}/|\mathbf{v}|$ of its straight-line motion.



The Distance from a Point to a Line in Space

Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$\boxed{d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}}$$

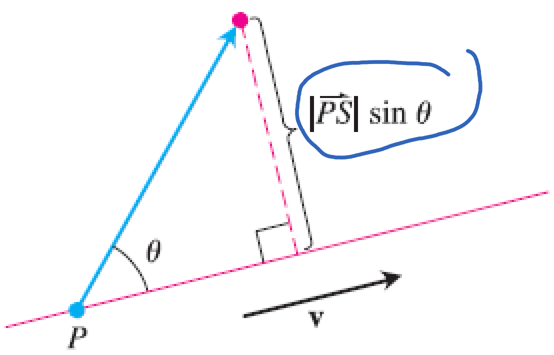


FIGURE 12.38 The distance from S to the line through P parallel to $\mathbf{v}$ is $|\overrightarrow{PS}| \sin\theta$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t. \quad , -\infty < t < \infty$$

$t = 0$

Sol.

$S(1,1,5)$, $P(1,3,0)$, $\vec{v} = i - j + 2k$

$\vec{PS} = -2j + 5k$

$$\vec{PS} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = i + 5j + 2k$$

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$$|\vec{PS} \times \vec{v}| = \sqrt{1+25+4} = \sqrt{30}$$

$$|\vec{v}| = \sqrt{1+1+4} = \sqrt{6}$$

$$\therefore d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

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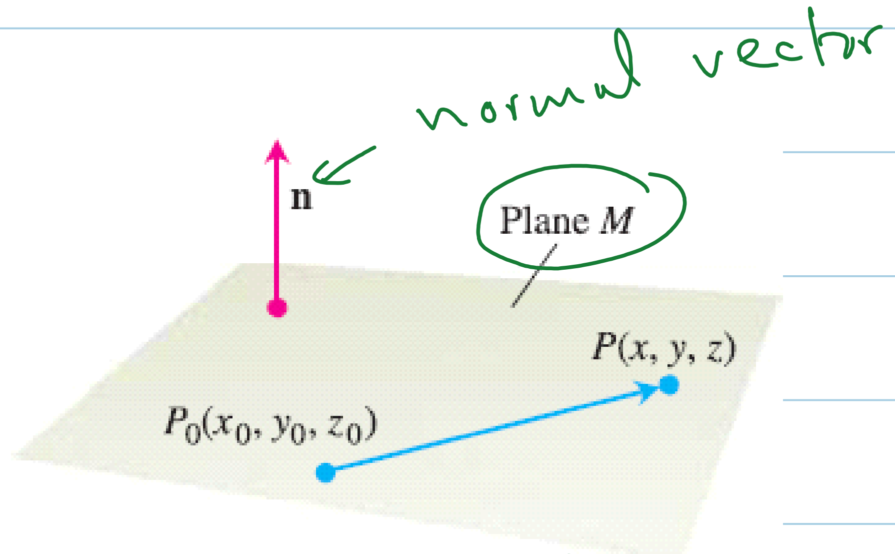
An Equation for a Plane in Space



$\overrightarrow{P_0P}$ is orthogonal to $\vec{n} = Ai+Bj+ck$

That is $\boxed{\overrightarrow{P_0P}\cdot\vec{n}=0}$ vector eq.

$$\left[(x-x_0)i+(y-y_0)j+(z-z_0)k\right]\cdot\left[Ai+Bj+Ck\right]=0$$

$\boxed{A(x-x_0)+B(y-y_0)+C(z-z_0)=0}$ Component eq.

$Ax+By+cz=\boxed{Ax_0+By_0+cz_0}$ Call it D

$\therefore \boxed{Ax+By+cz=D}$ is called

the Component eq. simplified for the plane passing through $P_0(x_0,y_0,z_0)$ normal to $\boxed{\vec{n}=Ai+Bj+ck}$.

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n}\cdot\overrightarrow{P_0P} = 0$ ✓

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ ✓

Component equation simplified: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

EXAMPLE 6 Find an equation for the plane through $P_0(\overset{x_0}{-3}, \overset{y_0}{0}, \overset{z_0}{7})$ perpendicular to $\boxed{\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}.}$

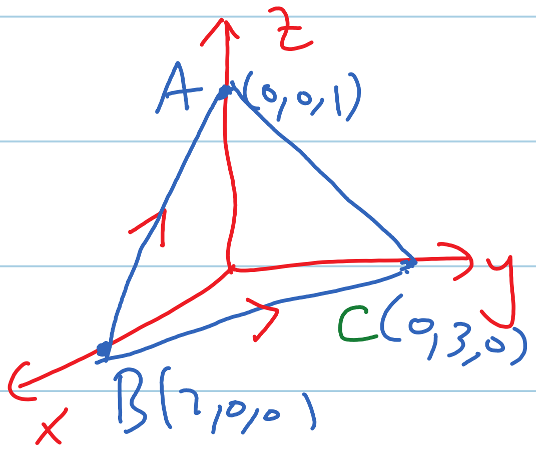
Sol.

$$5(x+3) + 2(y-0) - 1(z-7) = 0$$

$$5x + 15 + 2y - z + 7 = 0$$

$$\boxed{5x + 2y - z = -22}$$

EXAMPLE 7 Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.



Sol. $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$$= (2i - k) \times (3j - k)$$

$$= \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3i + 2j + 6k$$

You can take any point (Say $A(0,0,1)$)

The eq. for the plane is

$$3(x-0) + 2(y-0) + 6(z-1) = 0$$

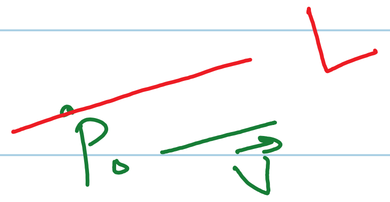
$$\boxed{3x + 2y + 6z = 6}$$

Lines of Intersection

Rmk. • Two planes are parallel iff their normals are parallel.

That is $M_1 \parallel M_2$ iff $\vec{n}_1 = k\vec{n}_2$, k scalar

• Two planes that are not parallel intersect in a line



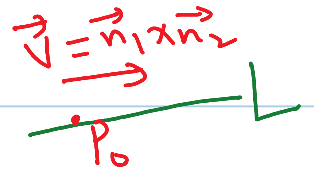
$\vec{?}$

EXAMPLE 8 Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

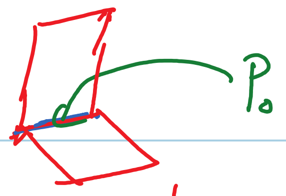
Sol. $\vec{n}_1 = 3i - 6j - 2k$, $\vec{n}_2 = 2i + j - 2k$

$$\vec{V} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 14i + 2j + 15k.$$



EXAMPLE 9 Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.



Sol. $\vec{V} = \vec{n}_1 \times \vec{n}_2 = 14i + 2j + 15k$ (see Ex8)

Put $z = 0$:

$$3x - 6y = 15$$
$$6\,(2x + y = 5$$
$$15x = 45 \Rightarrow \boxed{x = 3}$$
$$\Rightarrow 2(3) + y = 5$$
$$\boxed{y = -1}$$

$\therefore P_0(3, -1, 0)$

$$\therefore \vec{n} = 14i + 2j + 15k$$

$$P_0(3,-1,0).$$

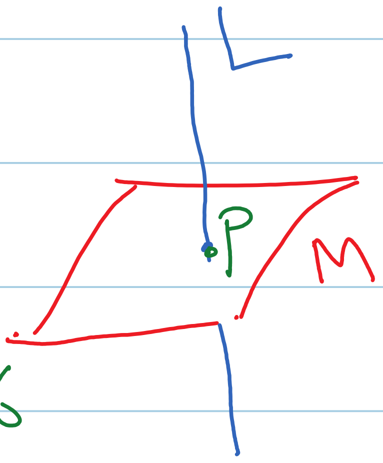
The line is

$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t, \quad -\infty < t < \infty$$

EXAMPLE 10 Find the point (x,y,z) where the line

$$\boxed{x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t}$$

intersects the plane $\boxed{3x + 2y + 6z = 6.}$



Solution. $3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6$

$$8 + \underbrace{6t} - \underbrace{4t} + 6 + \underbrace{6t} = 6$$

$$8t + 8 = 0 \Rightarrow \boxed{t = -1}$$

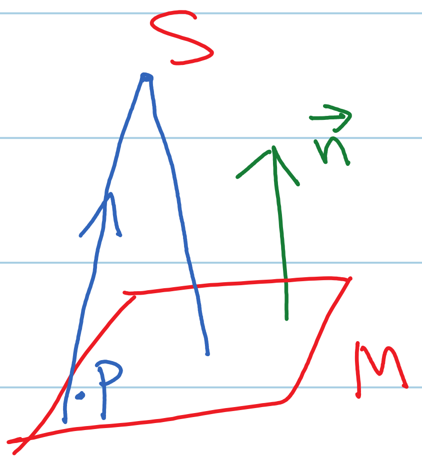
$$\therefore x = \frac{8}{3} + 2(-1) = \frac{2}{3}$$

$$y = -2(-1) = 2$$

$$z = 1 - 1 = 0$$

$\therefore$ the point is $P\left(\frac{2}{3}, 2, 0\right)$.

The Distance from a Point to a Plane



• The distance from S to the plane is

$$\boxed{d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|}$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.

EXAMPLE 11 Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Sol. $S(1,1,3)$, $\vec{n} = 3i + 2j + 6k$

$P(0,0,1)$ $\because$ put $x=y=0 \Rightarrow z=1$

$\overrightarrow{PS} = i + j + 2k$, $|\vec{n}| = \sqrt{9+4+36} = 7$

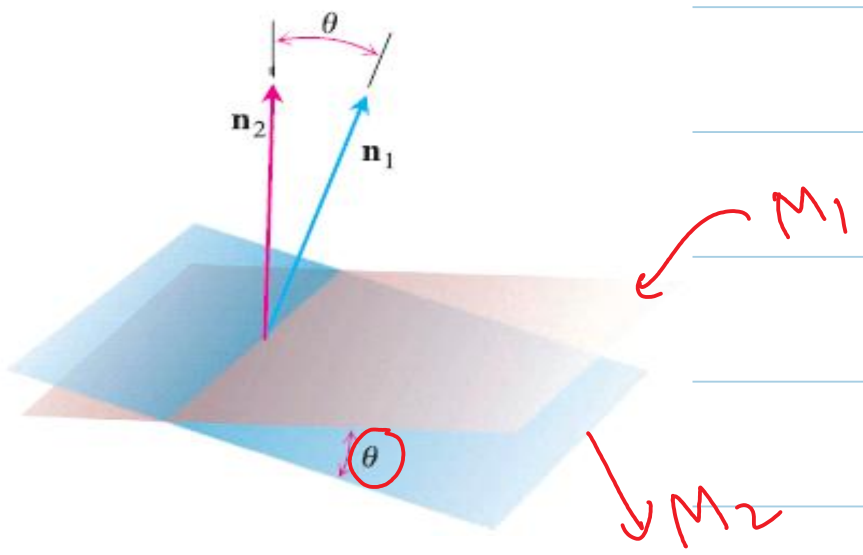
$\overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|} = (i+j+2k) \cdot \left(\frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k\right)$

$= \frac{3}{7} + \frac{2}{7} + \frac{12}{7} = \frac{17}{7}$

$\therefore$ distance $= \left|\overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|}\right| = \left|\frac{17}{7}\right| = \frac{17}{7}$

Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors (Figure 12.42).



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EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Sol. $\vec{n}_1 = 3i - 6j - 2k$, $\vec{n}_2 = 2i + j - 2k$

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{(3)(2) - 6(1) - 2(-2)}{\sqrt{9+36+4}\sqrt{4+1+4}}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right) \approx 1.38 \text{ radian}$$

$79°$

Summary 12.5

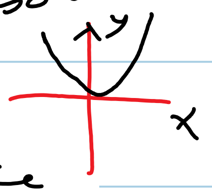
- line + line Segments in space.
- distance — point, Line; point, plane
- eq. of the plane in space.
- Intersections plane + plane → Line
  Plane + line → point
- Angles between planes is the angle between their normals.

$x^2+y^2=1$, z

Circular

12.6 Cylinders and Quadric Surfaces

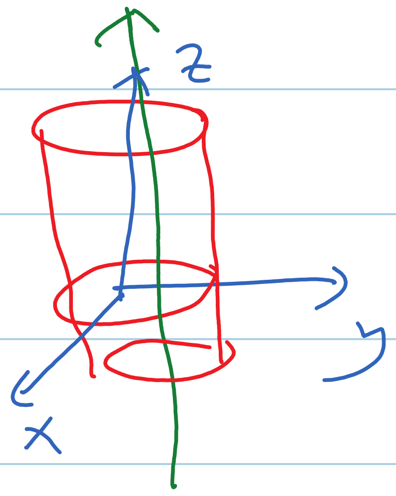
$y=x^2$ parabola



$y=x^2$, space

Cylinders

A **cylinder** is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a **generating curve** for the cylinder (Figure 12.43). In solid geometry, where *cylinder* means *circular cylinder*, the generating curves are circles, but now we allow generating curves of any kind. The cylinder in our first example is generated by a parabola.



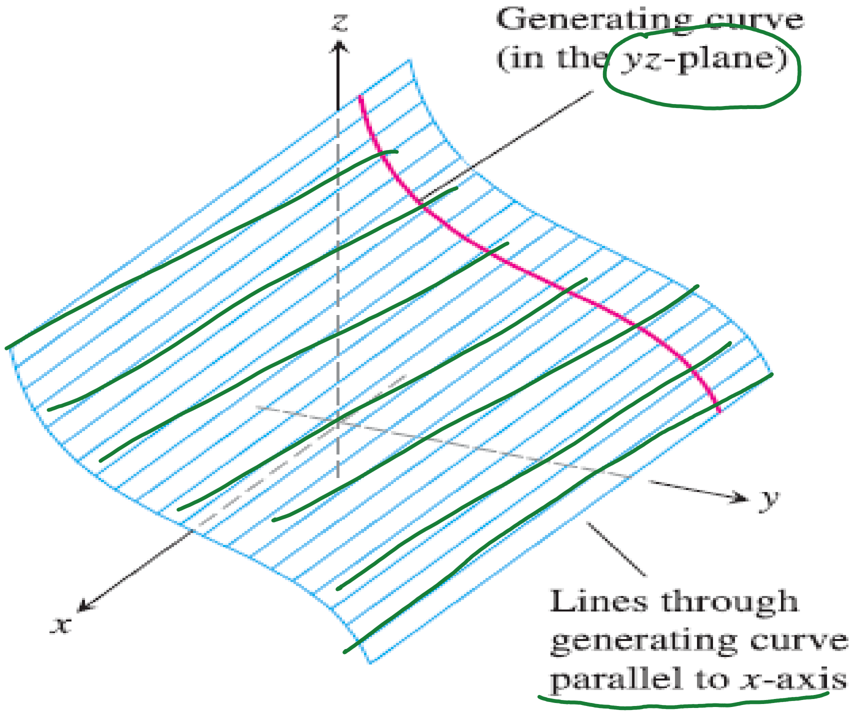
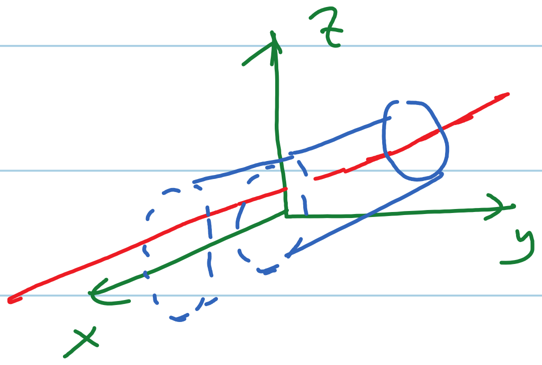


FIGURE 12.43 A cylinder and generating curve.

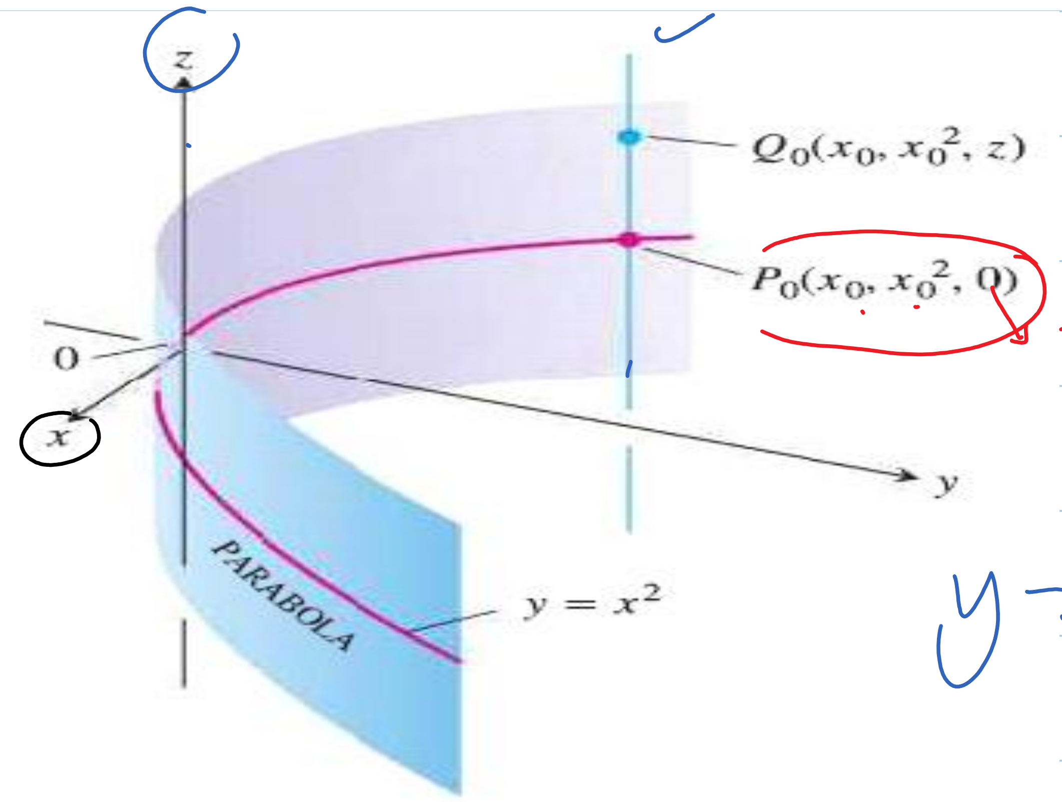


EXAMPLE 1 Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2$, $z = 0$ (Figure 12.44).

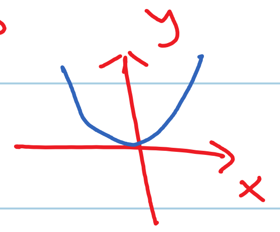
(x_0, x_0^2, z)

Solution The point $P_0(x_0, x_0^2, 0)$ lies on the parabola $y = x^2$ in the xy-plane. Then, for any value of z, the point $Q(x_0, x_0^2, z)$ lies on the cylinder because it lies on the line $x = x_0, y = x_0^2$ through P_0 parallel to the z-axis. Conversely, any point $Q(x_0, x_0^2, z)$ whose y-coordinate is the square of its x-coordinate lies on the cylinder because it lies on the line $x = x_0, y = x_0^2$ through P_0 parallel to the z-axis (Figure 12.44).

Regardless of the value of z, therefore, the points on the surface are the points whose coordinates satisfy the equation $y = x^2$. This makes $y = x^2$ an equation for the cylinder. Because of this, we call the cylinder "the cylinder $y = x^2$." ■



$z = 0$



$y = x^2$

$\frac{x^2}{4} + \frac{y^2}{4} = 1$

Quadric Surfaces

A **quadric surface** is the graph in space of a second-degree equation in x, y, and z. We focus on the special equation

$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

where A, B, C, D, and E are constants. The basic quadric surfaces are **ellipsoids**, **paraboloids**, **elliptical cones**, and **hyperboloids**. Spheres are special cases of ellipsoids. We present a few examples illustrating how to sketch a quadric surface, and then give a summary table of graphs of the basic types.

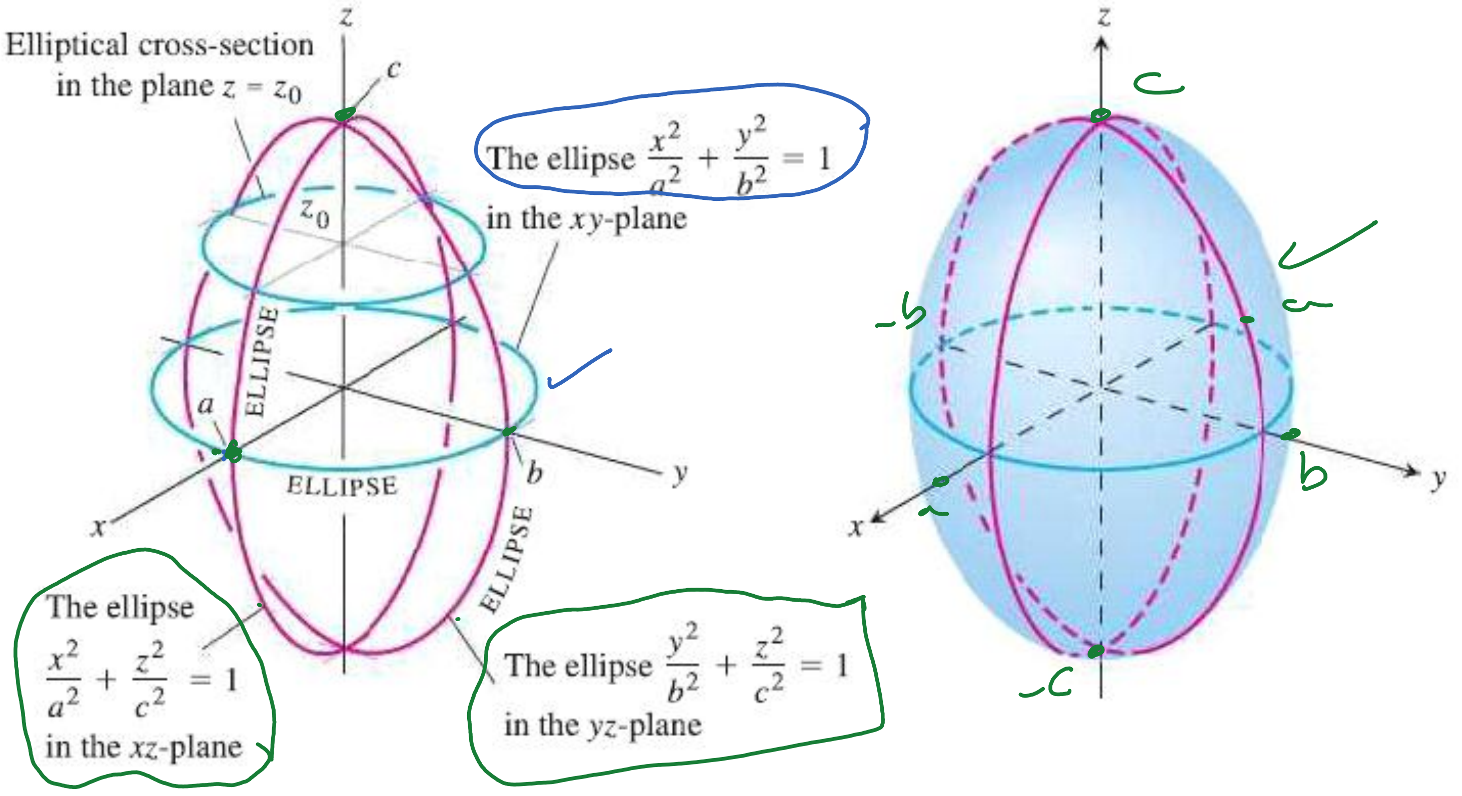
EXAMPLE 2 ① The **ellipsoid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$z = 0$ $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse in xy-plane

$x = 0$ $\quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ -- -- yz plane

$y = 0$ $\quad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$,, ,, xz-plane



a = b = c

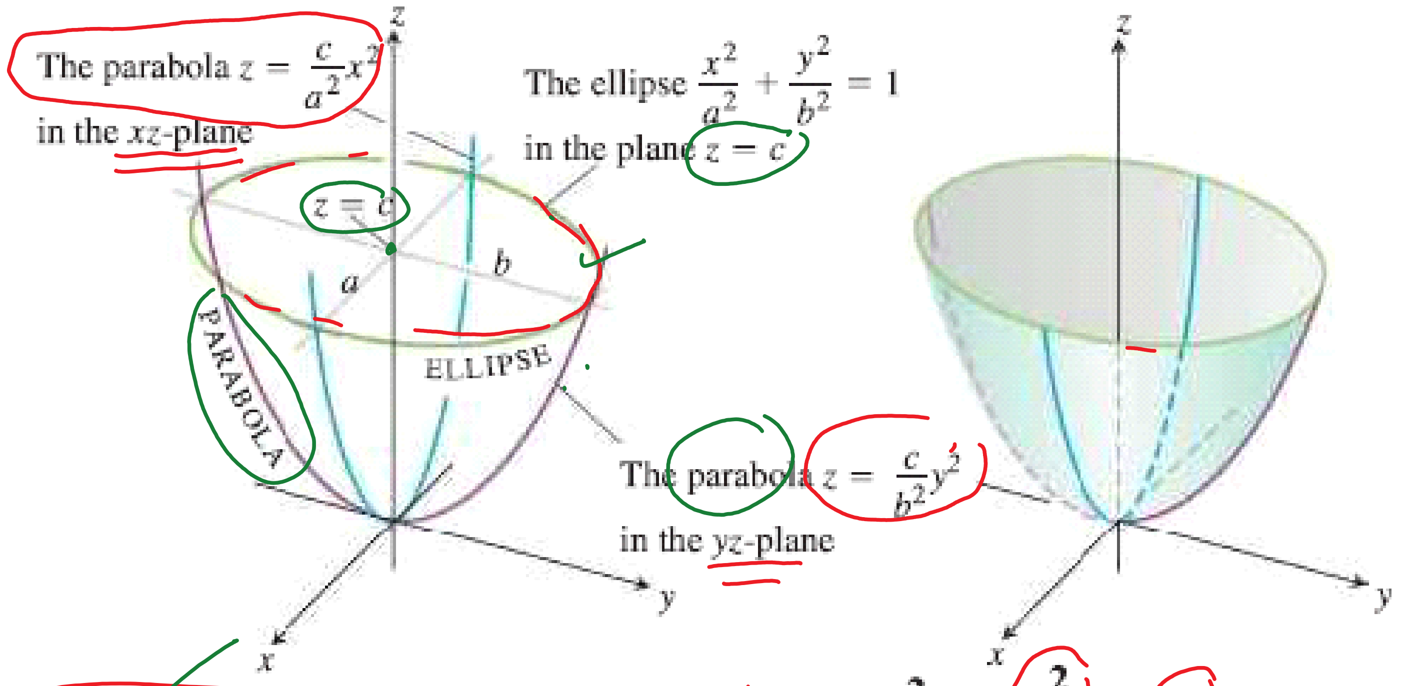
Spheres

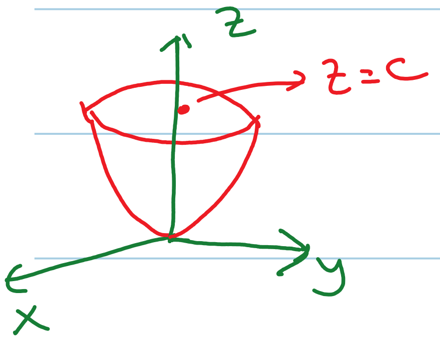
FIGURE 12.45 The ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

If any two of the semiaxes a, b, and c are equal, the surface is an **ellipsoid of revolution**. If all three are equal, the surface is a sphere. ■





② **ELLIPTICAL PARABOLOID** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$

y = 0

$x^2 = \frac{a^2}{c} z$

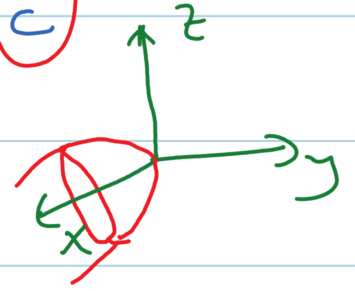
parabola

$z = c$: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

ellipse

ex. $\frac{y^2}{a^2}+\frac{z^2}{b^2}=\frac{x}{c}$

-2



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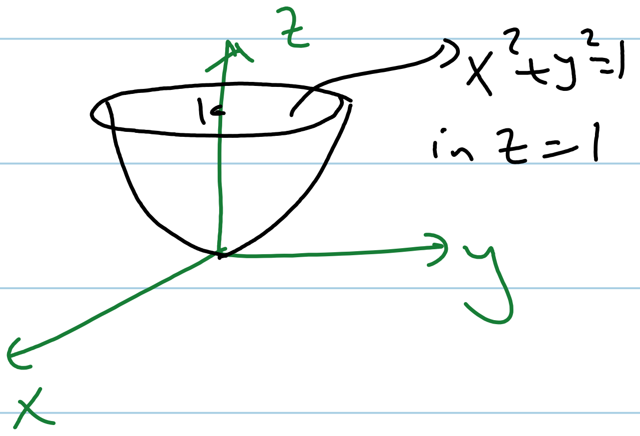
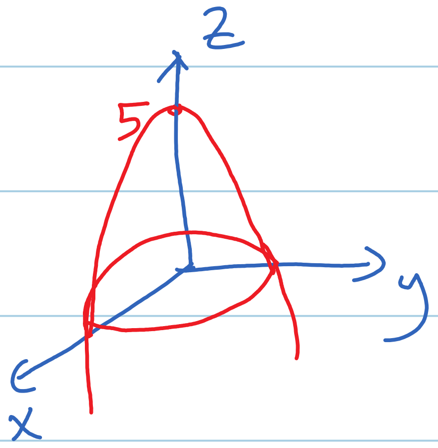
$x^2+y^2=z$

Circular paraboloid.

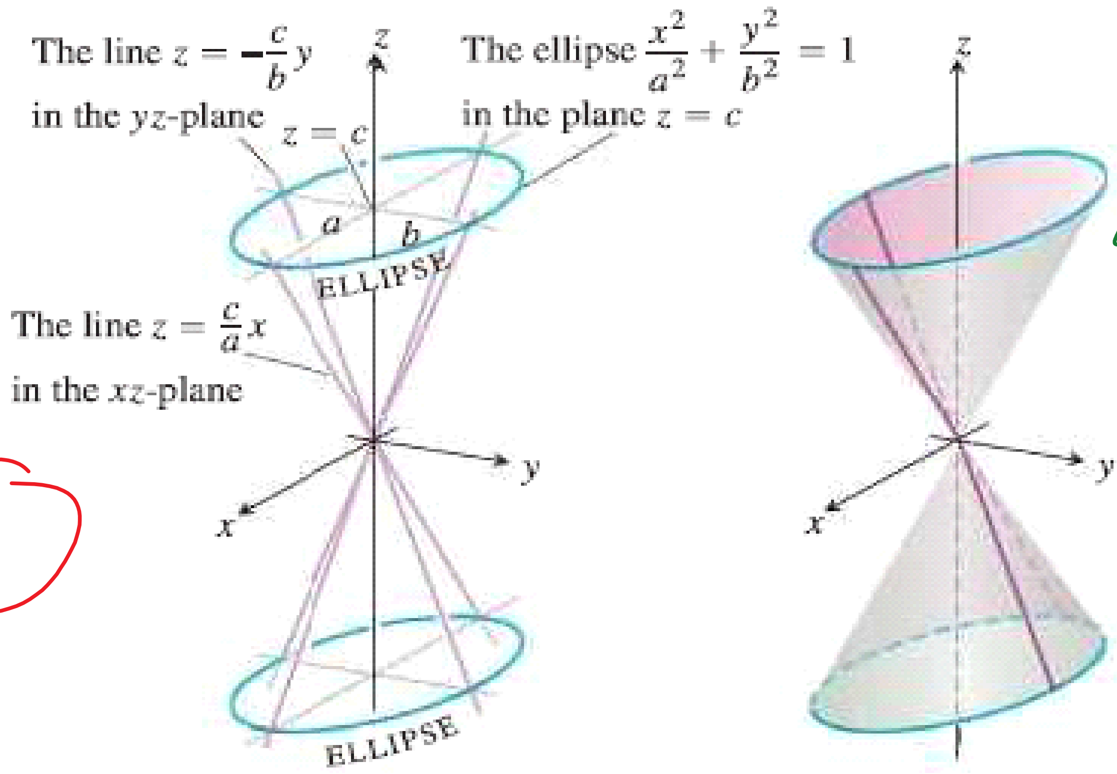


Sketch $z = 5-(x^2+y^2)$

$x^2+y^2=5-z$



③

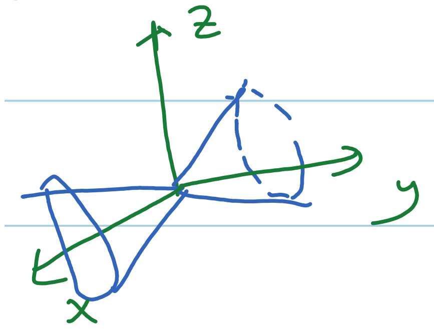


✓

ELLIPTICAL CONE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$

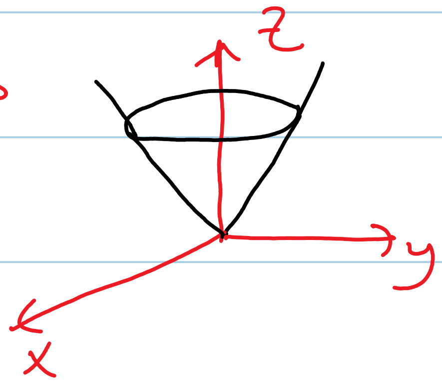
$y^2 \cancel{+x^2} = z$

$y^2+z^2=x^2$



ex. $x^2+y^2=z^2$ Circular cone

ex. sketch $z=\sqrt{x^2+y^2} \geq 0$



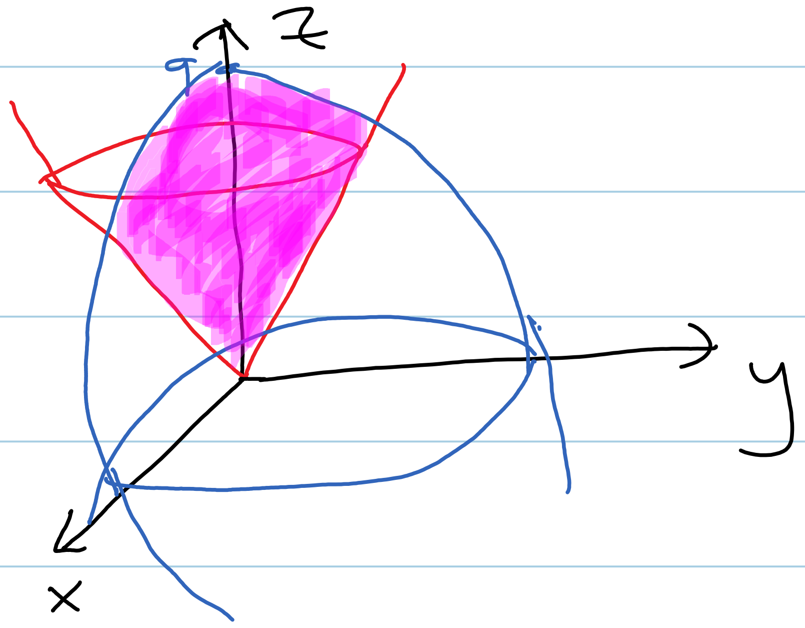
ex. $x^2+y^2=-z^2$

$x^2+y^2+z^2=0$

$\Rightarrow x=y=z=0$ $(0,0,0)$ Point (origin).

ex. $z=\sqrt{x^2+y^2}$ ✓ , $z=9-x^2-y^2$ ✓

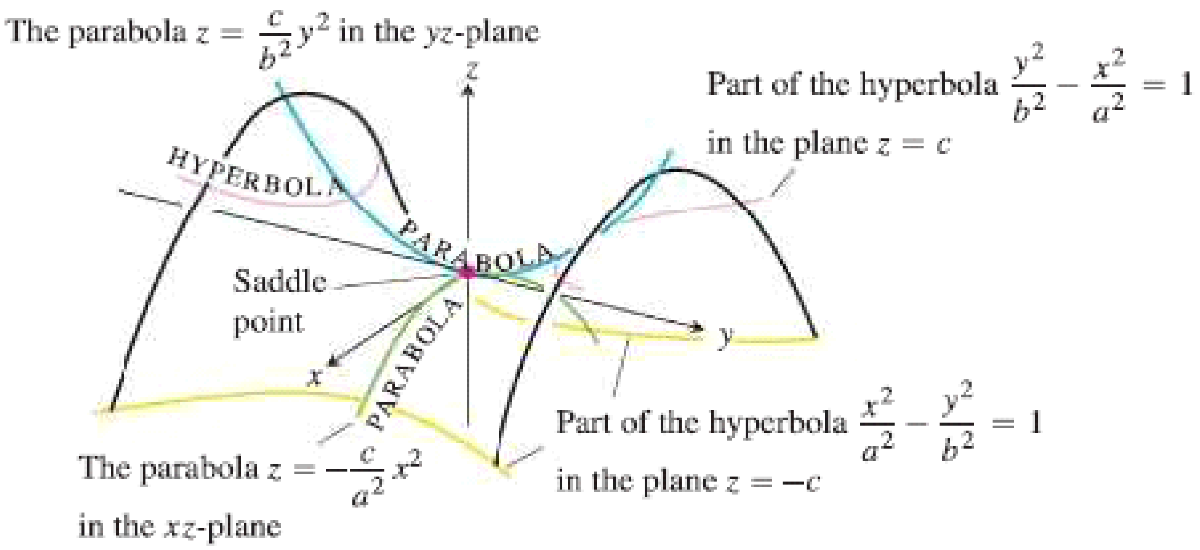
sketch the common region.

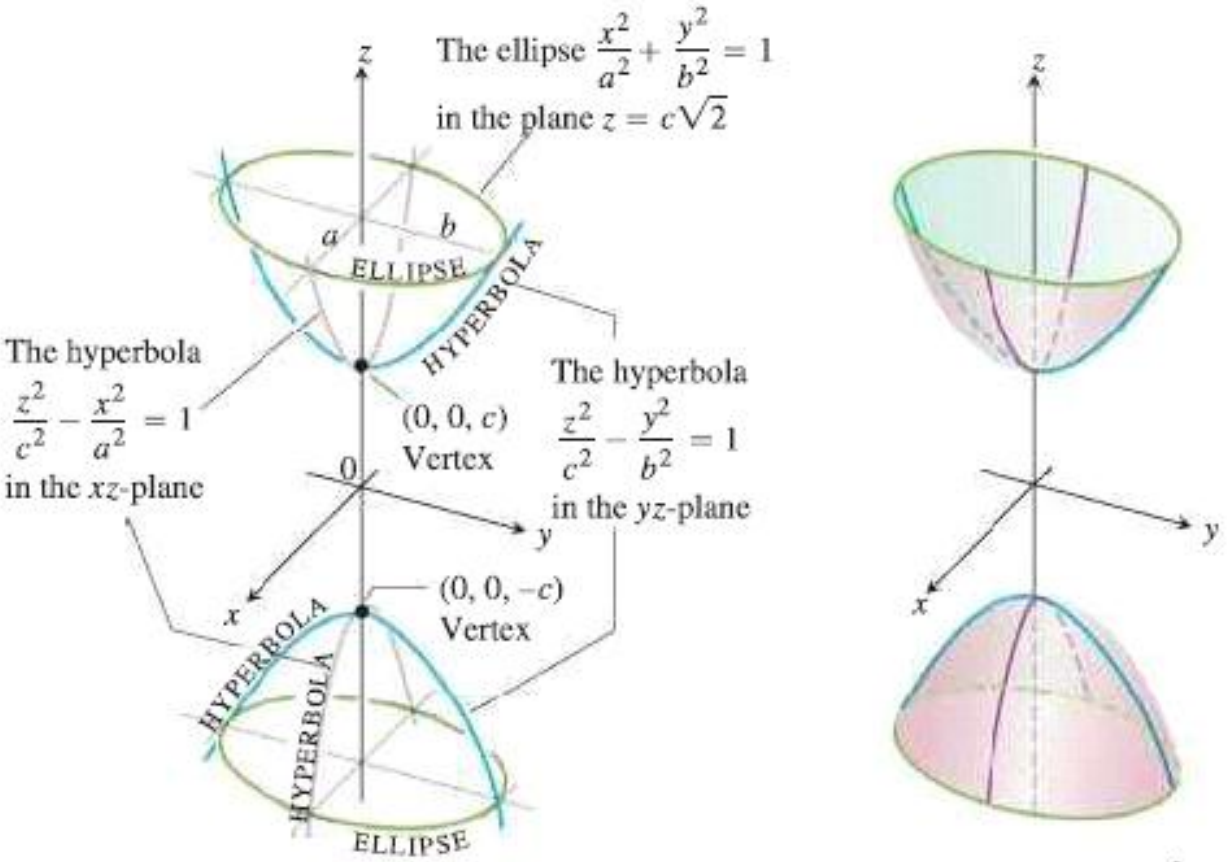


④

HYPERBOLOID OF ONE SHEET

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$





⑤

HYPERBOLOID OF TWO SHEETS

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

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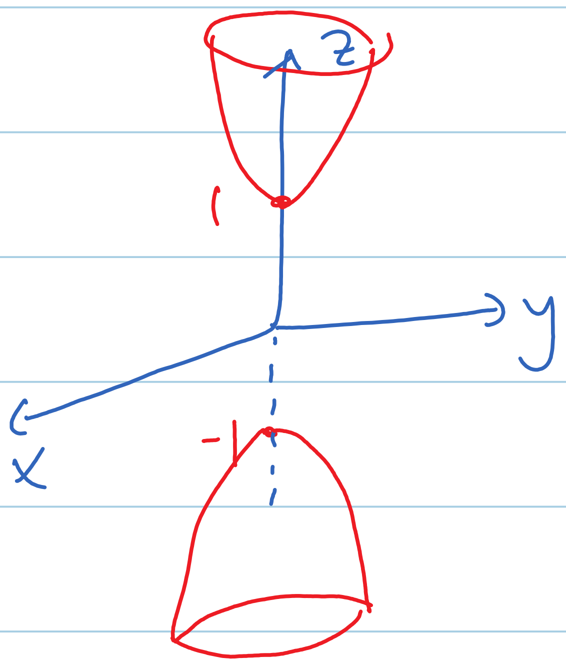
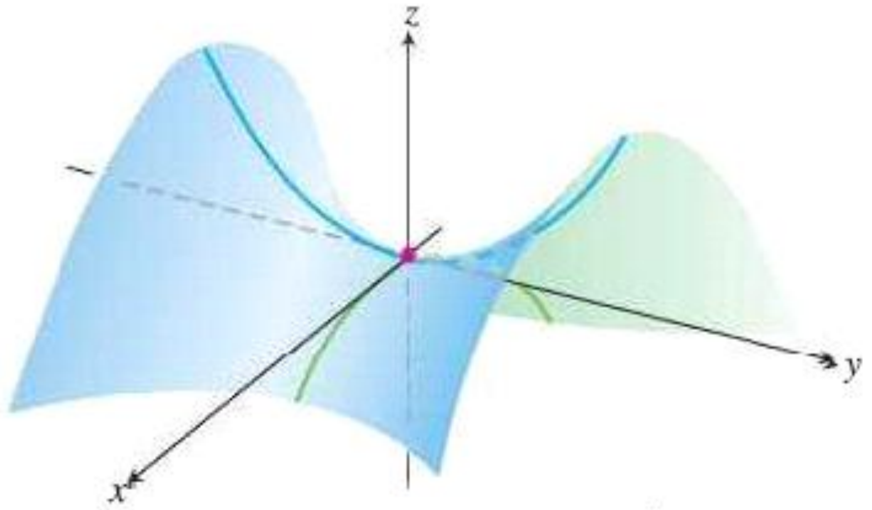
ex. $z^2 - x^2 - y^2 = 1$

$x = 0,\ y = 0 \Rightarrow z = \pm 1$



⑥



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$

Discussion (12.1 — 12.4)

12.1	660	8, 12, 14, 20, 26, 30, 36, 43, 56, 64,

20. **a.** $x^2 + y^2 \le 1, \quad z = 0$ **b.** $x^2 + y^2 \le 1, \quad z = 3$
c. $x^2 + y^2 \le 1$, no restriction on z

ⓐ the interior of the circle $x^2+y^2=1$
+ the boundary in the xy-plane

ⓑ 〃 〃 〃 〃 in the plane $z=3$.

ⓒ

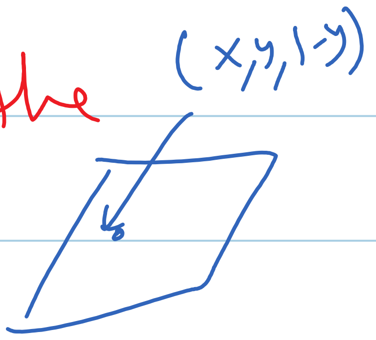
(c) A solid cylindrical column of radius 1 whose axis is the z-axis

36. The solid cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

Sol. $0 \le x \le 2, \quad 0 \le y \le 2, \quad 0 \le z \le 2$

24. **a.** $\boxed{z = 1 - y,}$ no restriction on x $(x, y, 1-y)$
b. $z = y^3, \quad x = 2$

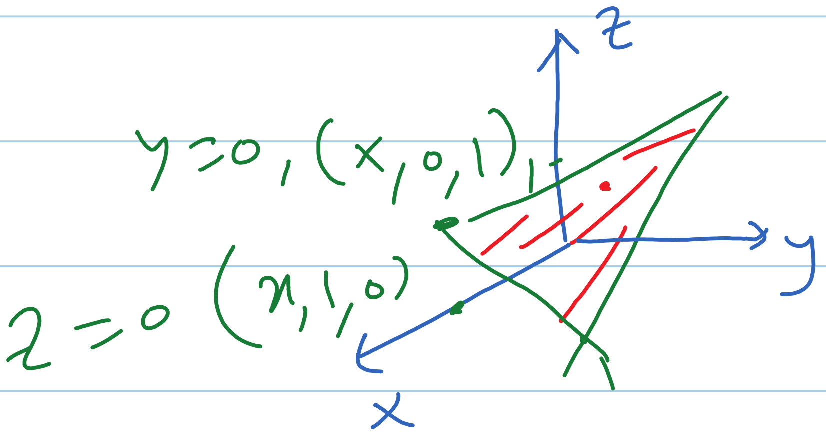
Sol. ⓐ All points that lie on the plane $z=1-y$.



$z+y=1$

ⓑ $(2, y, y^3)$

All points that lie
on the curve $z=y^3$
in the plane $x=2$



12.2 | 665 | 7, 10, 15, 18, 22, 25, 33, 40, 42

42. Linear combination Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, and $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Write $\boxed{\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2}$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is parallel to $\mathbf{w}$. (See Exercise 41.)

$\vec{u}_1 \parallel \vec{v}$ $\vec{u}_2 \parallel \vec{w}$.

$u_1 = a\vec{v}$ $\vec{u}_2 = b\vec{w}$

Sol:

$$\vec{u} = a\vec{v} + b\vec{w}$$

$$i - 2j = a(2i+3j) + b(i+j)$$

$$i - 2j = (2a+b)i + (3a+b)j$$

$$2a + b = 1 \quad — (A)$$

$$3a + b = -2 \quad — (B)$$

$$(B)-(A): \boxed{a = -3} \Rightarrow 2(-3) + b = 1$$

$$\boxed{b = 7}$$

$$\therefore \vec{u}_1 = a\vec{v} = -3(2i+3j)$$

$$= -6i - 9j$$

$$\vec{u}_2 = b\vec{w} = 7(i+j) = 7i + 7j$$

Notice $\vec{u}_1 \parallel \vec{v}$, $\vec{u}_2 \parallel \vec{w}$ and

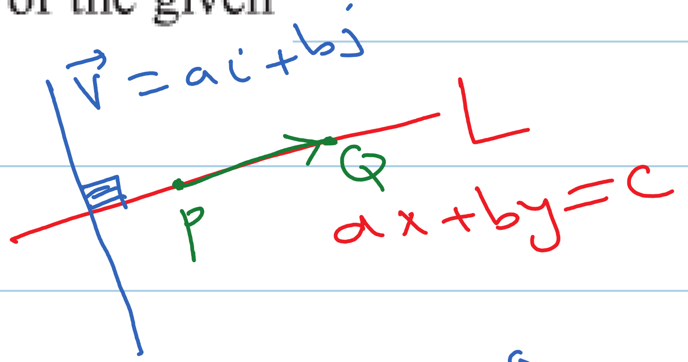
$$\vec{u}_1 + \vec{u}_2 = -6i - 9j + 7i + 7j$$

$$= i - 2j = \vec{u} .$$

12.3	674	5, 10, 17, 20, 31, 33, 45

31. Line perpendicular to a vector Show that $\boxed{\mathbf{v} = a\mathbf{i} + b\mathbf{j}}$ is perpendicular to the line $ax + by = c$ by establishing that the slope of the vector **v** is the negative reciprocal of the slope of the given line.

$$\vec{PQ} \cdot \vec{V} = 0$$



Case1 $b \neq 0$

$$ax+by=c \Rightarrow by = -ax + c$$

$\text{slope} = -\frac{a}{b}$

$$\boxed{y = -\frac{a}{b}x + \frac{c}{b}}, \quad b \neq 0$$

$$P\left(x_1, -\frac{a}{b}x_1 + \frac{c}{b}\right), \quad Q\left(x_2, -\frac{a}{b}x_2 + \frac{c}{b}\right)$$

$$\vec{PQ} = (x_2 - x_1)\,i + \left(-\frac{a}{b}x_2 + \frac{a}{b}x_1\right)j$$

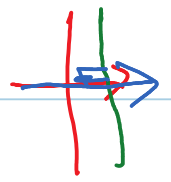
$$\boxed{\vec{PQ} = (x_2 - x_1)\,i - \frac{a}{b}(x_2 - x_1)\,j}$$

$$\boxed{\vec{V} = ai + bj}$$

$$\vec{PQ} \cdot \vec{V} = a(x_2 - x_1) - \frac{a}{b}(x_2 - x_1)(b)$$

$$= a(x_2 - x_1) - a(x_2 - x_1) = 0$$

Case 2 $\boxed{b=0}$ $\boxed{\vec{V} = ai}$, $\boxed{ax = c}$



$\vec{V}$ is perpendicular to the vertical line $ax=c$

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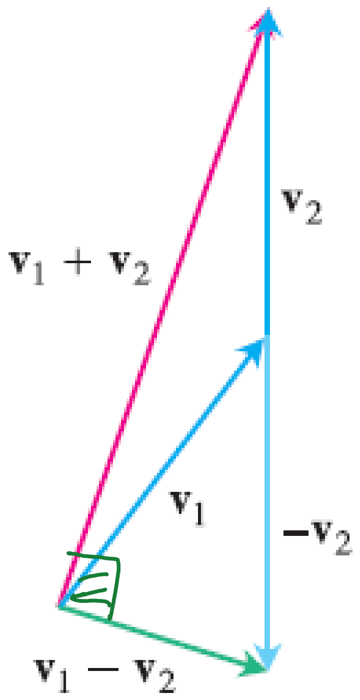
17. Sums and differences In the accompanying figure, it looks as if $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are orthogonal. Is this mere coincidence, or are there circumstances under which we may expect the sum of two vectors to be orthogonal to their difference? Give reasons for your answer.



$$(\vec{V}_1+\vec{V}_2)\cdot(\vec{V}_1-\vec{V}_2)$$

$$= \vec{V}_1\cdot\vec{V}_1 - \cancel{\vec{V}_1\cdot\vec{V}_2} + \cancel{\vec{V}_2\cdot\vec{V}_1} - \vec{V}_2\cdot\vec{V}_2$$

$$= |V_1|^2 - |V_2|^2$$

$$= 0 \quad \text{if} \quad |V_1| = |V_2|.$$

The sum of two vectors of equal length is *always* orthogonal to their difference.

$\vec{u}+\vec{v} \perp \vec{u}-\vec{v}$ if $|\vec{u}| = |\vec{v}|$.

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12.4	682	3, 16, 20, 23, 27, 34, 40, 45

34. Double cancellation If $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

$ax = ay$

$x = y$

$\cancel{a}b = \cancel{a}c, a \neq 0$

Yes.

proof.

$\vec{u} \times \vec{v} = \vec{u} \times \vec{w} \Rightarrow \vec{u} \times (\vec{v} - \vec{w}) = \vec{0}$ — ①

$\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} \Rightarrow \vec{u} \cdot (\vec{v} - \vec{w}) = 0$ — ②

Suppose $\boxed{\vec{v} \neq \vec{w}}$. Then

Eq ① $\Rightarrow \vec{v} - \vec{w} \parallel \vec{u}$

$\boxed{\vec{v} - \vec{w} = \alpha \vec{u}}$ — ③, $\alpha \neq 0$ scalar

put ③ into ②

$\vec{u} \cdot (\alpha \vec{u}) = 0$

$\alpha (\vec{u} \cdot \vec{u}) = 0$

$\alpha |\vec{u}|^2 = 0 \quad , \alpha \neq 0$

$\Rightarrow \vec{u} = \vec{0}$ contradiction.

$\therefore \vec{v} = \vec{w}$

13

VECTOR-VALUED FUNCTIONS AND MOTION IN SPACE

13.1 Curves in Space and Their Tangents

$\vec{r}(t) = \overrightarrow{OP}$

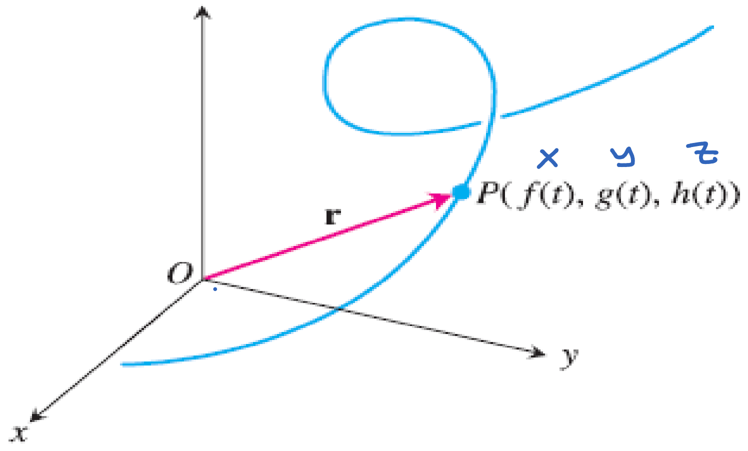


FIGURE 13.1 The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

When a particle P moves through the space during a time t ∈ I "interval", then the coordinates of this particle defined on I as

$\boxed{x = f(t),\ y = g(t),\ z = h(t),\ t \in I}$ ①

the set C of all points $(x,y,z) = (f(t), g(t), h(t))$, $t \in I$ is called a space curve.

- eq ① is called parametric eqs of C and t is called a parameter.
- A curve in space can be also represented in vector form: $\vec{r}(t) = \overrightarrow{OP}$

$$\boxed{\vec{r}(t) = f(t)i + g(t)j + h(t)k}$$

is the vector from (0,0,0) to P(f(t), g(t), h(t)) at time t is called the particle's position vector.

- f, g, h are called the component functions of $\vec{r}(t)$

Df. (vector function).

A vector function or a vector valued function is a function whose domain is a set of real numbers and whose range is a set of vectors.

$$\vec{r}(t) = f(t)\,i + g(t)\,j + h(t)\,k$$

$$\underset{\in \mathbb{R}}{t} \longmapsto \underbrace{\langle f(t), g(t), h(t) \rangle}_{\in \text{ vectors.}}$$

- Real valued functions are called scalar functions.
- The components of $\vec{r}$ are scalar functions of t.
- the domain of a vector function is the common domain of its components

i.e., $\text{Dom}(\vec{r}(t)) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$.

Ex: If $\boxed{\vec{r}(t) = t^3 i + \ln(3-t)\,j + \sqrt{t}\,k}$

Find ⓐ $\vec{r}(2)$ ⓑ Domain $(\vec{r}(t))$.

Sol. $\vec{r}(2) = 8i + \ln(3-2)\,j + \sqrt{2}\,k$

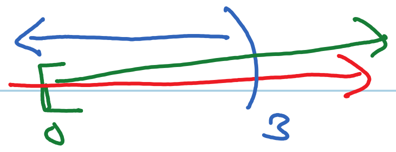
$= 8i + \sqrt{2}\,k$

ⓑ $f(t) = t^3$, $g(t) = \ln(3-t)$, $h(t) = \sqrt{t}$

$D_f = (-\infty, \infty)$, $D_g : 3-t > 0 \Rightarrow t < 3$

$D_g = (-\infty, 3)$

$D_h : t \geq 0 \Rightarrow [0, \infty) = D_h$



$\therefore D_{\vec{r}(t)} = [0, 3)$

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EXAMPLE 1 Graph the vector function

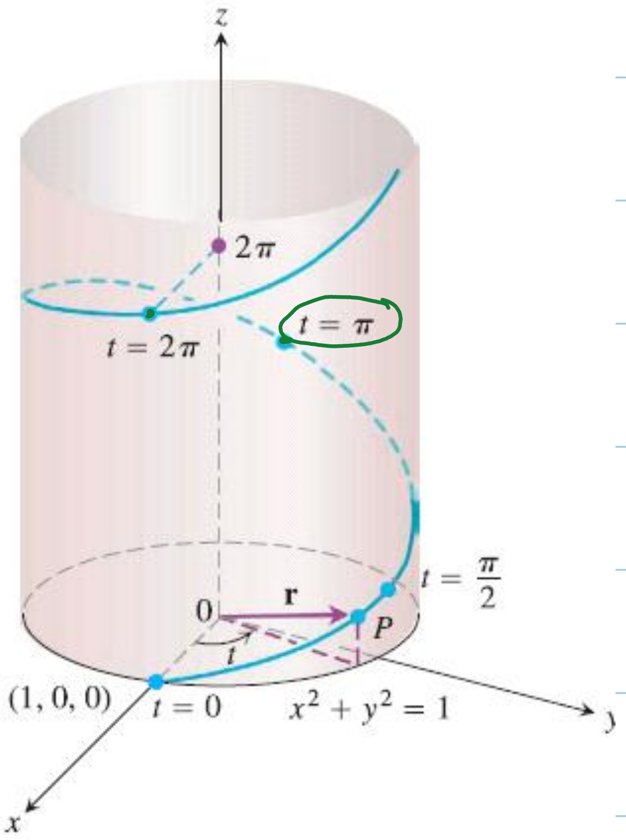
$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

Solution $x(t) = \text{Cost}$, $y(t) = \text{Sint}$, $z(t) = t$,
$-\infty < t < \infty$

$$x^2(t) + y^2(t) = \cos^2 t + \sin^2 t = 1$$

$t = 0$: $\vec{r}(0) = i$

$t = \frac{\pi}{2}$ $\vec{r}\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2}\, i + \sin\frac{\pi}{2}\, j + \frac{\pi}{2} k$
$= j + \frac{\pi}{2} k.$



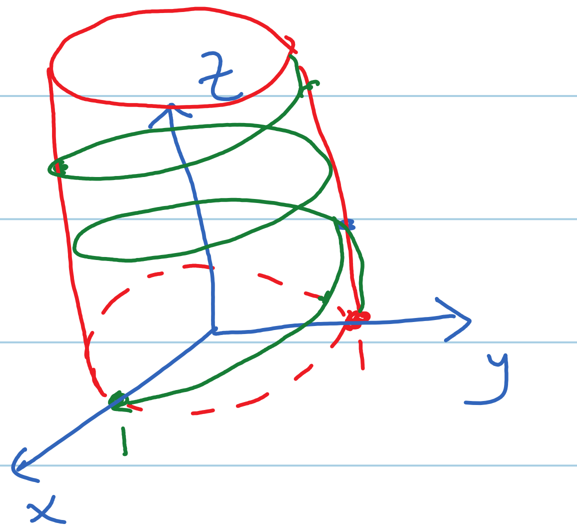
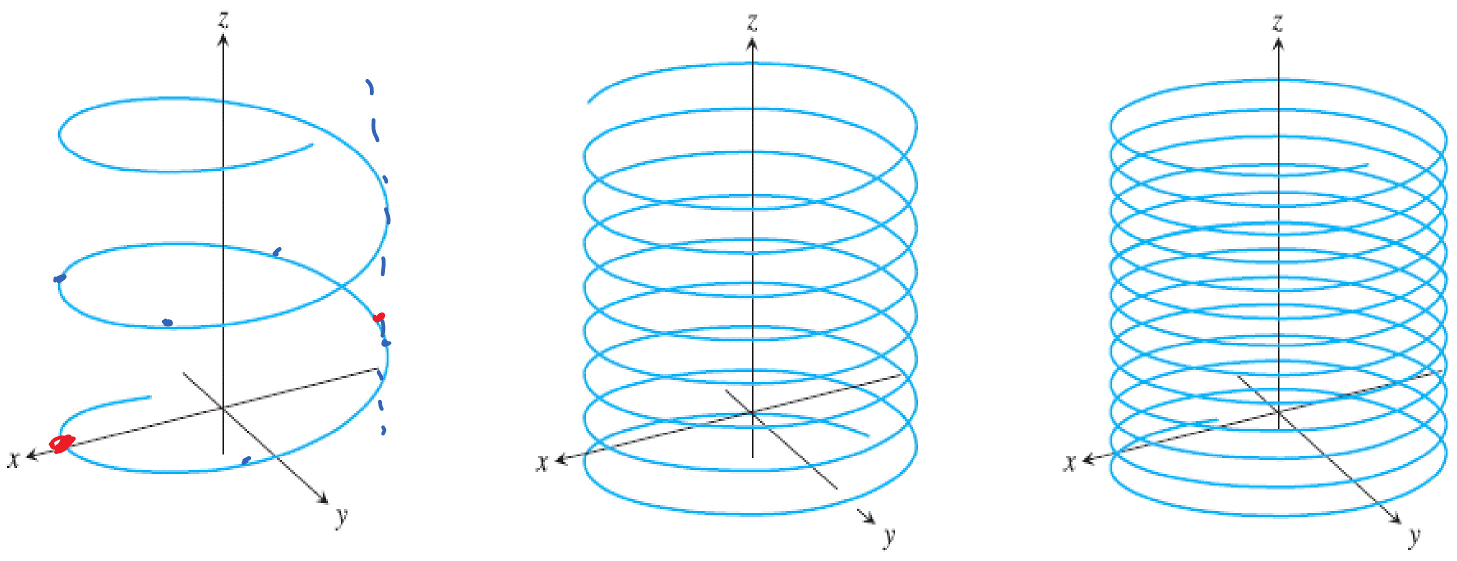


FIGURE 13.3 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

Spiral.



$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$ $\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$

FIGURE 13.4 Helices spiral upward around a cylinder, like coiled springs.

Limits and Continuity

$$\vec{r}(t) = x(t)\,i + y(t)\,j + z(t)\,k$$

$$\lim_{t\to t_0} \vec{r}(t) = \left(\lim_{t\to t_0} x(t)\right) i + \left(\lim_{t\to t_0} y(t)\right) j + \left(\lim_{t\to t_0} z(t)\right) k$$

provided the limits of the components exist.

ex. $$\vec{r}(t) = \overbrace{\left(\frac{t^2-1}{t-1}\right)}^{x(t)} i + \overbrace{\frac{\sin t}{t}}^{y(t)} j + \overset{z(t)}{(\ln t)} k$$

find $\lim_{t\to 1} \vec{r}(t)$

Sol. $$\lim_{t\to 1} x(t) = \lim_{t\to 1} \frac{t^2-1}{t-1} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{t\to 1} \frac{2t}{1} = \boxed{2}$$

$$\lim_{t\to 1} y(t) = \lim_{t\to 1} \frac{\sin t}{t} = \boxed{\sin 1}.$$

$$\lim_{t\to 1} z(t) = \lim_{t\to 1} \ln t = \ln 1 = \boxed{0}$$

$$\therefore \lim_{t\to 1} \vec{r}(t) = 2\,i + (\sin 1)\,j$$

Ex. $$\vec{r}(t) = (1+t^3)\,i + t e^{-t}\,j + \frac{\sin t}{t}\,k$$

Find $\lim_{t\to 0} \vec{r}(t)$

Sol. $$\lim_{t\to 0} \vec{r}(t) = \left[\lim_{t\to 0} (1+t^3)\right] i + \left(\lim_{t\to 0} t e^{-t}\right) j$$

$$+\left(\lim_{t\to 0}\frac{\sin t}{t}\right)k$$

$$= i + k.$$

Df (continuity)

. A vector function $\vec{r}(t)$ is continuous at $t = t_0$ in its domain if

$$\lim_{t\to t_0}\vec{r}(t) = \vec{r}(t_0).$$

. The vector function is cont. on D if it is cont. at every points in D.

<u>ex.</u> Where $\vec{r}(t) = (\cos t)\, i + (\sin t)\, j + t k$ is continuous?

<u>Ans:</u> cont. on $(-\infty, \infty)$.

<u>Ex.</u> $\vec{r}_{(t)} = (\cos t)\, i + (\sin t)\, j + \lfloor t \rfloor k$

is cont. on $(-\infty, \infty) \setminus \{0, \pm 1, \pm 2, \pm 3, \dots\}$

تذكر/

$$\lfloor t \rfloor = \begin{cases} 1, & 1 \le t < 2 \\ 0, & 0 \le t < 1 \\ -1, & -1 \le t < 0 \\ \vdots & \end{cases}$$

أكبر عدد صحيح greatest integer function

Derivatives and Motion

DEFINITION The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at** t if f, g, and h have derivatives at t. The derivative is the vector function

$$\vec{\mathbf{r}}'(t) = \boxed{\frac{d\mathbf{r}}{dt}} = \underbrace{\lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}}_{\text{تعريف المشتقة}} = \overset{\checkmark}{\frac{df}{dt}}\mathbf{i} + \overset{\checkmark}{\frac{dg}{dt}}\mathbf{j} + \overset{\checkmark}{\frac{dh}{dt}}\mathbf{k}.$$

$$= f'(t)\,i + g'(t)\,j + h'(t)\,k$$

Ex. $\vec{r}(t) = (\ln t)\,i + \frac{t+1}{t+2}\,j + (t\ln t)\,k$

Find $\frac{d\vec{r}}{dt}$ at $t = 1$.

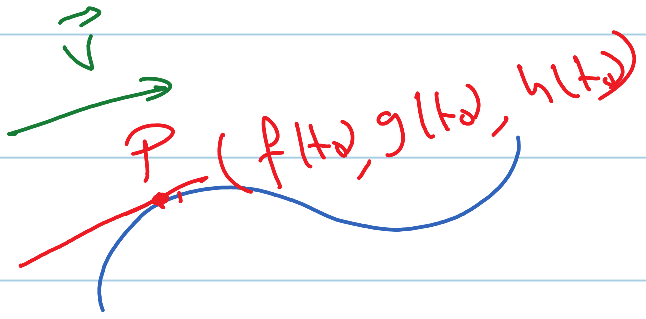
Sol. $$\frac{d\vec{r}}{dt} = \frac{d}{dt}(\ln t)\,i + \frac{d}{dt}\left(\frac{t+1}{t+2}\right)j + \frac{d}{dt}(t\ln t)\,k$$

$$= \frac{1}{t}\,i + \frac{(t+2)(1) - (t+1)(1)}{(t+2)^2}\,j + \left(t\cdot\frac{1}{t} + \ln t\cdot 1\right)k$$

$$= \frac{1}{t}\,i + \frac{1}{(t+2)^2}\,j + (1+\ln t)\,k$$

At $t = 1$, $\frac{d\vec{r}}{dt} = i + \frac{1}{9}\,j + k$

Rmk. The tangent line to the curve at a point $P(f(t_0), g(t_0), h(t_0))$ is defined to be the line through P parallel to $\vec{v} = \frac{d\vec{r}}{dt}$ at $t = t_0$



Ex. Find parametric eqs for the line tangent to the curve

فكرة
12.5

$$\vec{r}(t) = (\ln t)\,i + \left(\frac{t+1}{t+2}\right)j + (t\ln t)\,k$$

at $t_0 = 1$

Sol. $\vec{r}(1) = \ln 1\, i + \frac{2}{3}\,j + \ln 1\, k$ $P\left(0, \frac{2}{3}, 0\right)$

$(0, \frac{2}{3}, 0)$.

$$\vec{v} = \frac{d\vec{r}}{dt}\Big|_{t=1} = i + \frac{1}{9}j + k$$

متجه المماس

$\therefore$ Parametric eqs

$$x = 0 + t = t$$
$$y = \frac{2}{3} + \frac{1}{9}t \quad , \quad -\infty < t < \infty.$$
$$z = 0 + t = t$$

Rmk. ① A vector function $\vec{r}(t)$ is diffble if it is diffble at every point of its domain.

② The curve traced by $\vec{r}(t)$ is Smooth if $\frac{d\vec{r}}{dt}$ is continuous and $\frac{d\vec{r}}{dt} \neq \vec{0}$.

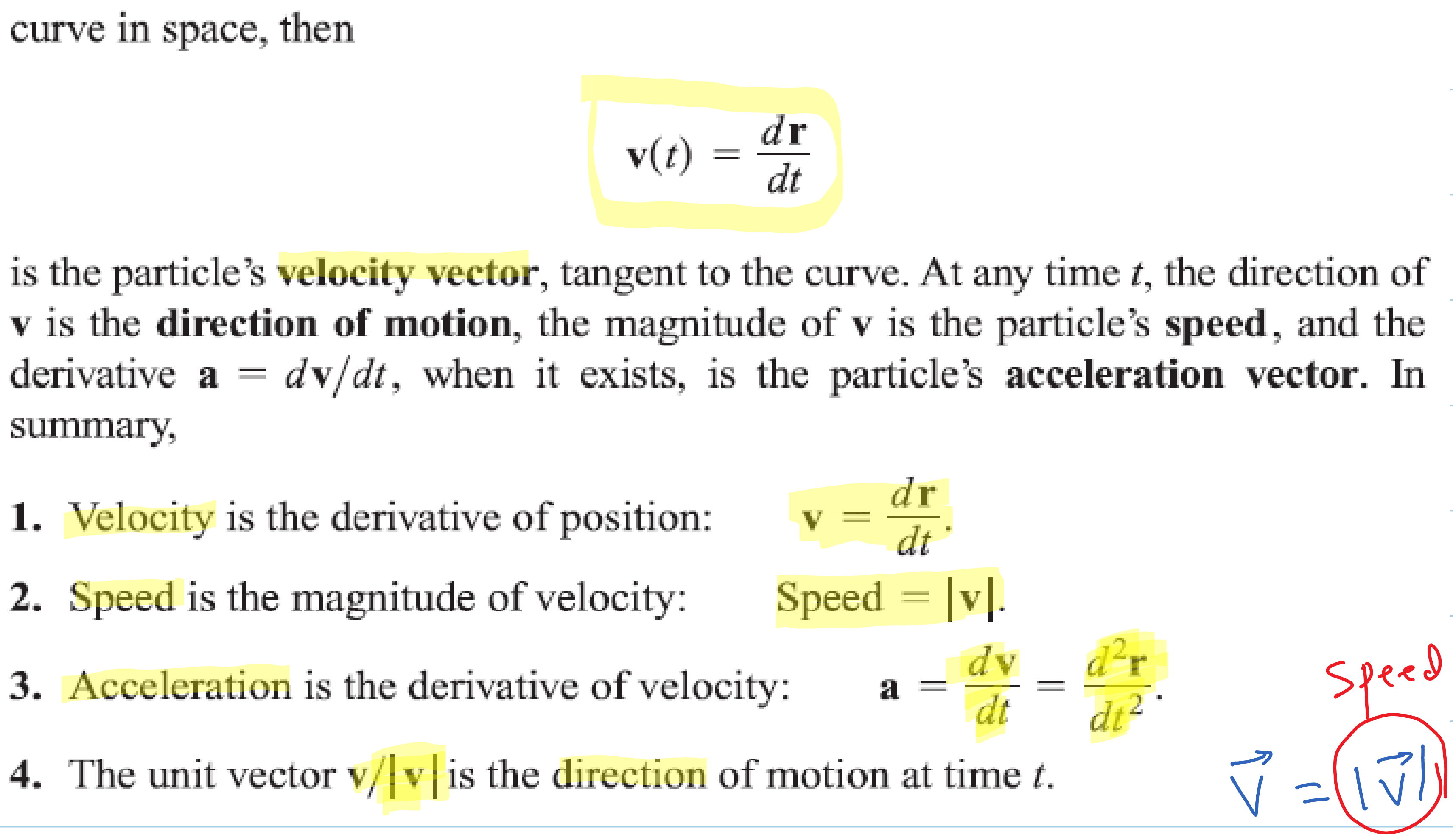
③ The curve is called piece-wise smooth if it is made up of a finite number of smooth curves pieced together in continuous fashion.

Ex. C:

C_1, C_2, C_3 are Smooth Curves

but the curve C is piecewise smooth.

$\frac{d\vec{r}}{dt}$ is cont. & $\neq 0$



DEFINITIONS If **r** is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of **v** is the **direction of motion**, the magnitude of **v** is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \frac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

Speed

$\vec{V} = \boxed{|\vec{V}|} \cdot \boxed{\frac{\vec{V}}{|\vec{V}|}}$

direction

Ex. If $\boxed{\vec{r}(t) = e^{-t} i + (2\cos 3t) j + (2\sin 3t) k}$ is the position vector. Find the velocity, speed, acceleration, direction at $t=0$.

Sol. $\boxed{\vec{V}(t) = \frac{d\vec{r}}{dt} = -e^{-t} i - (6\sin 3t) j + (6\cos 3t) k}$

$\vec{V}(0) = -i + 6k$.

Speed at $t=0$ is $|\vec{V}(0)| = \sqrt{1+36} = \sqrt{37}$

$\vec{a}(t) = \frac{d\vec{V}}{dt} = e^{-t} i - (18\cos 3t) j - (18\sin 3t) k$

$\vec{a}(0) = i - 18j$

$\vec{V}(0) = (\text{Speed})\ (\text{direction})$

$$= |\vec{V}(0)| \quad \frac{\vec{V}(0)}{|\vec{V}(0)|}$$

$$= \sqrt{37}\left(-\frac{1}{\sqrt{37}} i + \frac{6}{\sqrt{37}} k \right).$$

Ex. $\vec{r}(t) = 4\cos(\frac{t}{2})\, i + (4\sin\frac{t}{2})\, j$

Find:

ⓐ $\vec{V}(\pi)$ and $\vec{a}(\pi)$

ⓑ Sketch them as vectors on the curve.

ⓒ the angle between $\vec{V}(\pi)$ and $\vec{a}(\pi)$.

Sol. ⓐ $\vec{V}(t) = \frac{d\vec{r}}{dt} = -2\sin(\frac{t}{2})\, i + 2\cos(\frac{t}{2})\, j$

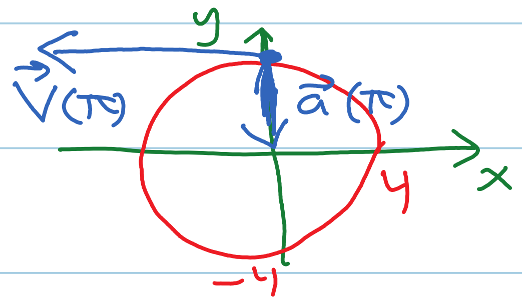
$\vec{a}(t) = \frac{d\vec{V}}{dt} = -\cos(\frac{t}{2})\, i - \sin(\frac{t}{2})\, j$

$\vec{V}(\pi) = -2i$, $\vec{a}(\pi) = -j$

ⓑ $x(t) = 4\cos(\frac{t}{2})$, $y = 4\sin(\frac{t}{2})$

$$x^2 + y^2 = 16\Big(\underbrace{\cos^2\frac{t}{2} + \sin^2\frac{t}{2}}_{1} \Big) = 16$$

$\therefore x^2 + y^2 = 16$



At $t = \pi$, $x = 0$, $y = 4$ $(0,4)$

ⓒ Angle between $\vec{V}(\pi)$ and $\vec{a}(\pi)$

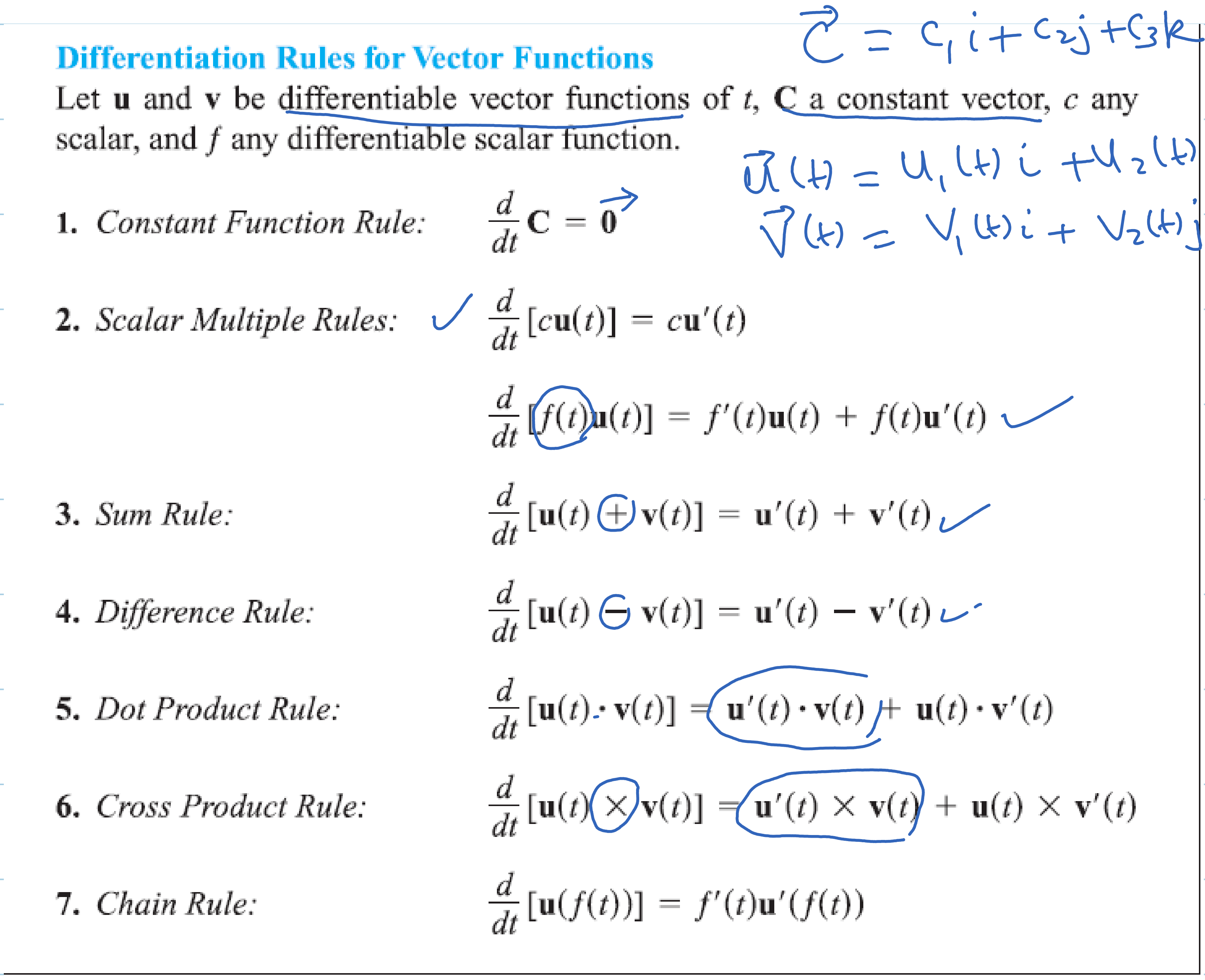
Sol. $\vec{V}(\pi) = -2i$, $\vec{a}(\pi) = -j$

$$\theta = \cos^{-1}\left(\frac{\vec{V}(\pi)\cdot\vec{a}(\pi)}{|\vec{V}(\pi)|\,|\vec{a}(\pi)|}\right)$$

$$= \cos^{-1}\left(\frac{-2(0)+0(-1)}{(2)(1)}\right)$$

$$= \cos^{-1}(0) = \pi/2 .$$

Differentiation Rules



$\vec{C} = c_1 i + c_2 j + c_3 k$

$\vec{u}(t) = u_1(t)\,i + u_2(t)\,j + u_3(t)\,k$

$\vec{V}(t) = V_1(t)\,i + V_2(t)\,j + V_3(t)\,k$

Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of t, **C** a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \vec{0}$
2. *Scalar Multiple Rules:* ✓ $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ ✓
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$ ✓
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$ ✓
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

$|\vec{r}(t)| = \text{Constant}$

Ex. If **r** is a differentiable vector function of t of constant length, then

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0.$$

Sol. $|\vec{r}(t)| = k$

$|\vec{r}(t)|^2 = k^2$, k is constant.

$\vec{r}(t) \cdot \vec{r}(t) = k^2$ $\vec{v} \cdot \vec{v} = |\vec{v}|^2$

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = \frac{d}{dt}(k^2)$$

$$\vec{r} \cdot \frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt} \cdot \vec{r} = 0$$

$$2\left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) = 0$$

$$\Rightarrow \vec{r} \cdot \frac{d\vec{r}}{dt} = 0$$ ∎

ex. $\vec{r}(t) = \cos t \, i + \sin t \, j$, $|\vec{r}(t)| = \sqrt{\cos^2 t + \sin^2 t}$
$= 1$ constant

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = (\cos t \, i + \sin t \, j) \cdot (-\sin t \, i + \cos t \, j) = 0$$

Ex. The converse of the last example is true

If $\boxed{\vec{r} \cdot \frac{d\vec{r}}{dt} = 0}$, then $|\vec{r}(t)| =$ constant.

Proof. $|\vec{r}|^2 = \vec{r} \cdot \vec{r}$

$$\frac{d}{dt}|\vec{r}|^2 = 2\vec{r} \cdot \frac{d\vec{r}}{dt} = 2(0) = 0$$ given.

$$\Rightarrow \frac{d}{dt}|\vec{r}|^2 = 0$$

$|\vec{r}(t)|^2 =$ Constant

$|\vec{r}(t)| =$ Con.stant.

13.2 Integrals of Vector Functions; ~~Projectile Motion~~

Integrals of Vector Functions

DEFINITION The **indefinite integral** of **r** with respect to t is the set of all antiderivatives of **r**, denoted by $\int \mathbf{r}(t)\,dt$. If **R** is any antiderivative of **r**, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \boxed{\mathbf{C}.} \rightarrow \text{Constant vector.}$$

Ex. $\int ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt$

$$= \left(\int \cos t\,dt\right) i + \left(\int 1\,dt\right) j - \left(\int 2t\,dt\right) k$$

$$= (\sin t + c_1) i + (t + c_2) j - (t^2 + c_3) k$$

$$= (\sin t) i + t j - t^2 k + \vec{C}, \text{ where}$$

$$\vec{C} = c_1 i + c_2 j + c_3 k$$

ex. $\int \left[\left(\frac{2}{\sqrt{1-t^2}}\right) i + \left(\frac{\sqrt{3}}{1+t^2}\right) k\right] dt$

$$= 2\int \frac{1}{\sqrt{1-t^2}}\,dt \; i + \sqrt{3}\int \frac{1}{1+t^2}\,dt \; k$$

$$= (2 \sin^{-1} t)\, i + (\sqrt{3} \tan^{-1} t)\, k.$$

DEFINITION If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is **r**, and the **definite integral** of **r** from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

Ex. $\int_0^{\pi/3} \left[(\sec t \tan t)\, i + (\tan t)\, j + (2\sin t \cos t)\, k\right] dt$

$$= \left(\int_0^{\pi/3} \sec t \tan t \, dt\right) i + \left(-\int_0^{\pi/3} \underbrace{\frac{-\sin t}{\cos t}\, dt}\right) j + \int_0^{\pi/3} \sin(2t)\, dt \Big) k$$

$\int \frac{u'}{u} du = \ln|u| + c$

$$= \sec t \Big|_0^{\pi/3} i + -\ln|\cos t| \Big|_0^{\pi/3} + -\frac{\cos(2t)}{2} \Big|_0^{\pi/3} k$$

$$= (2-1)\, i + \left(-\ln\left(\tfrac{1}{2}\right) + \ln 1\right) j + \left(\frac{\frac{1}{2}}{2} + \frac{1}{2}\right) k$$

$$= i + (\ln 2)\, j + \frac{3}{4} k .$$

Q10) $I = \int_0^{\frac{\pi}{4}} \left(\sec t \; i + \tan^2 t \; j - t \sin t \; k\right) dt$

$$I_1 = \int_0^{\frac{\pi}{4}} \sec t \, dt = \ln|\sec t + \tan t| \Big|_0^{\pi/4}$$

$$= \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \cancel{\ln|\sec 0 + \tan 0|} \xrightarrow{} 0$$

$$= \boxed{\ln(\sqrt{2} + 1)}$$

$$I_2 = \int_0^{\pi/4} \tan^2 t \, dt = \int_0^{\pi/4} (\sec^2 t - 1)\, dt$$

$$= (\tan t - t) \Big|_0^{\pi/4}$$

$$= \left(\tan\tfrac{\pi}{4} - \tfrac{\pi}{4}\right) - (0 - 0)$$

$$= \boxed{1 - \frac{\pi}{4}}$$

$$I_3 = \int_0^{\pi/3} t \sin t \, dt$$

f & its deriv. g & its integrals

$$\begin{tikzcd} t \arrow[rd, "(+)"] & \sin t \\ 1 \arrow[rd, "(-)"] & -\cos t \\ 0 & -\sin t \end{tikzcd}$$

$$I_3 = \left(-t\cos t + \sin t\right)\Big|_0^{\pi/4}$$

$$= \left(-\frac{\pi}{4}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) - (0 + 0)$$

$$= -\frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2} = \frac{(4-\pi)\sqrt{2}}{8}$$

$$\therefore I = I_1 i + I_2 j + I_3 k$$

$$= \ln(1+\sqrt{2})\, i + \left(1 - \frac{\pi}{4}\right) j + \left(\frac{4-\pi}{8}\right)\sqrt{2}\, k$$

12. Differential equation: $\frac{d\mathbf{r}}{dt} = (180t)\mathbf{i} + (180t - 16t^2)\mathbf{j}$

Initial condition: $\mathbf{r}(0) = 100\mathbf{j}$

Find $\vec{r}$

$$\vec{r}(t) = \int \left[(180t)\, i + (180t - 16t^2)\, j\right] dt$$

$$\vec{r}(t) = 90t^2\, i + \left(90t^2 - \frac{16}{3}t^3\right) j + \vec{C}$$

$$\vec{r}(0) = 0\, i + (0 - 0)\, j + \vec{C} = 100j$$

$$\therefore \vec{r}(t) = (90t^2)\, i + \left(90t^2 - \frac{16}{3}t^3 + 100\right) j \qquad \therefore \boxed{\vec{C} = 100j}$$

13.3 Arc Length in Space

$r' = x' i + y' j + z' k = V(t)$

Arc Length Along a Space Curve

DEFINITION The **length** of a smooth curve $\boxed{\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \boxed{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}\, dt. \qquad \boxed{|V(t)|} \text{ Speed}$$

Arc Length Formula

$$L = \int_a^b |\mathbf{v}|\, dt$$

⑦ Ex. Find the length of the curve

$$\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \left(2\sqrt{2}/3\right)t^{3/2}\mathbf{k}, \quad 0 \le t \le \pi$$

Sol.

$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad z(t) = \frac{2\sqrt{2}}{3}t^{3/2}$$

$$\frac{dx}{dt} = \cos t - t\sin t, \quad \frac{dy}{dt} = \sin t + t\cos t$$

$$\frac{dz}{dt} = \frac{2\sqrt{2}}{3}\cdot\frac{3}{2}t^{\frac{1}{2}} = \sqrt{2}\,t^{\frac{1}{2}}.$$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$= (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + \left(\sqrt{2}\,t^{\frac{1}{2}}\right)^2$$

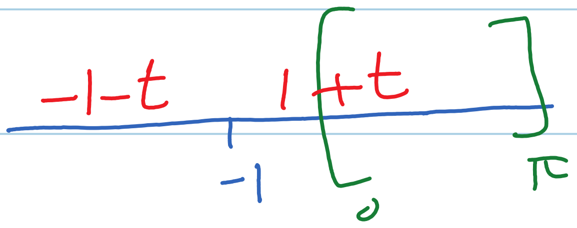
$$= \underbrace{\cos^2 t} - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \underbrace{\sin^2 t}$$
$$+ \cancel{2t\sin t\cos t} + t^2\cos^2 t + 2t$$

$$= \cos^2 t + \sin^2 t + t^2(\sin^2 t + \cos^2 t) + 2t$$

$$= 1 + t^2 + 2t = (1+t)^2$$

$$\therefore |V(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(1+t)^2}$$
$$= |1+t|$$

$$\therefore L = \int_0^{\pi} |v(t)|\, dt = \int_0^{\pi} |\overset{\geq 0}{1+t}|\, dt$$



$$= \int_0^{\pi} (1+t)\, dt$$

$$= \left(t + \frac{t^2}{2}\right)\Big|_0^{\pi}$$

$$= \pi + \frac{\pi^2}{2}$$

Unit Tangent Vector

$$\vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$\vec{T}$ is called a unit tangent vector to the smooth curve $\left(\frac{d\vec{r}}{dt} \neq 0\right)$

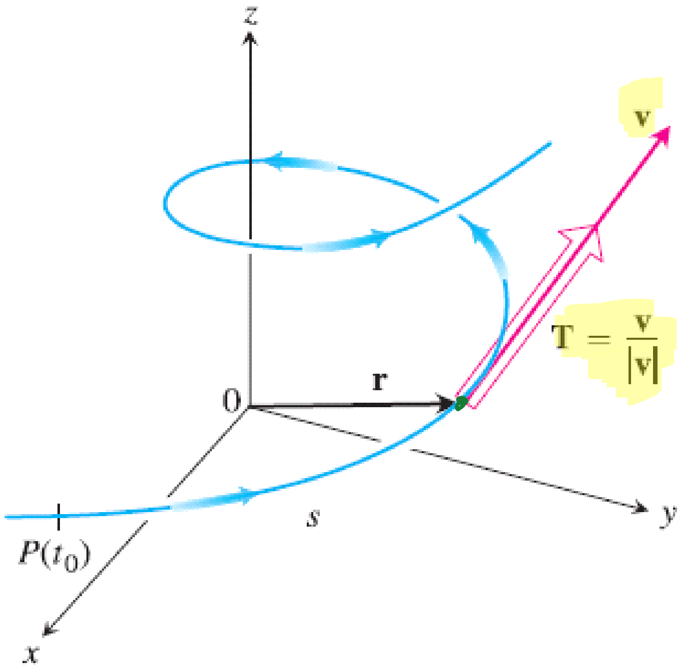


FIGURE 13.15 We find the unit tangent vector **T** by dividing **v** by |**v**|.

EXAMPLE 3 Find the $\overset{\vec{T}}{\text{unit tangent vector}}$ of the curve

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$

Sol. $\vec{v}(t) = \frac{d\vec{r}}{dt} = (-3\sin t)i + (3\cos t)j + (2t)k.$

$$|\vec{v}(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$$

$$= \sqrt{9(\sin^2 t + \cos^2 t) + 4t^2}$$

$$= \sqrt{9 + 4t^2}$$

$$\vec{T} = \frac{\vec{V}(t)}{|\vec{V}(t)|}$$

$$\vec{T} = \frac{-3\sin t}{\sqrt{9+4t^2}}\, i + \frac{3\cos t}{\sqrt{9+4t^2}}\, j + \frac{2t}{\sqrt{9+4t^2}}\, k$$

Arc Length Parameter with Base Point $\boxed{P(t_0)}$ a to b

$$\boxed{s(t)} = \int_{\boxed{t_0}}^{\boxed{t}} \boxed{\sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}}\, d\tau = \int_{t_0}^{t} \underline{|\mathbf{v}(\tau)|}\, d\tau$$

$$s(b) = L = \int_a^{\boxed{b}} |v(t)|\, dt$$

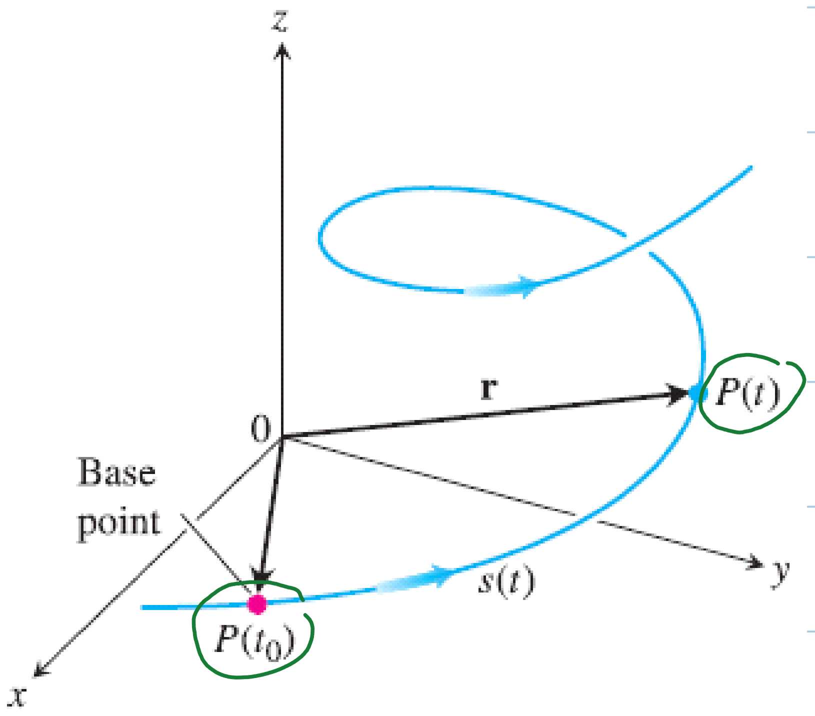


FIGURE 13.14 The directed distance along the curve from $P(t_0)$ to any point $P(t)$ is

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau .$$

Ex. Consider $\boxed{\vec{r}(t) = (4\cos t)\, i + (4\sin t)\, j + 3t\, k}$

(a) Find the arc length parameter with base $P(0)$ on $\boxed{0} \le t \le \frac{\pi}{4}$.

(b) Use (a), to find the length of the curve on $\frac{\pi}{2} \le t \le \pi$.

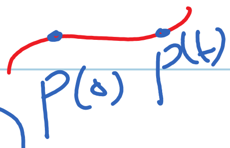
Sol. (a) $S(t) = \int_0^t |V(\tau)|\, d\tau$.

$$V(t) = \vec{r}\,'(t) = (-4\sin t)\, i + (4\cos t)\, j + 3k$$

$$|\vec{V}(t)| = \sqrt{16\sin^2 t + 16\cos^2 t + 9}$$

$$= \sqrt{16(1)+9} = \sqrt{25} = 5$$

$$\therefore \quad S(t) = \int_0^t |V(\tau)|\, d\tau = \int_0^t 5 \, d\tau$$



$$= 5\tau \Big|_0^t = 5t$$

$$\therefore \quad \boxed{S(t) = 5t}$$

$t=0 \longrightarrow t=t$

Rmk. length from $t=0$ to $t=\frac{\pi}{4}$ is

$$S\left(\frac{\pi}{4}\right) = 5\frac{\pi}{4}.$$

ⓑ the length of the curve on

$\frac{\pi}{2} \leq t \leq \pi$ is

$$S(\pi) - S\left(\frac{\pi}{2}\right) = 5\pi - 5\frac{\pi}{2}$$

$$= 5\frac{\pi}{2}.$$

Rmk. $S(t) = \int_{t_0}^{t} |V(\tau)|\, d\tau$

$$\left(\int_a^x f(t)\,dt\right)' = f(x).$$

$$\boxed{\frac{dS}{dt} = |V(t)|}$$

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \overset{\vec{V}(t)}{\frac{d\vec{r}}{dt}} \cdot \frac{1}{|V(t)|} = \vec{T}$$

$$\therefore \boxed{\vec{T} = \frac{\vec{V}}{|\vec{V}|} \underline{\underline{\text{or}}} \ (*) \ \frac{d\vec{r}}{ds}} \checkmark \quad , \quad s(t) = \int_{t_0}^{t} |V(\tau)| d\tau$$

arc length parameter with base $p(t_0)$.

Ex. $\vec{r}(t) = \cos t\, i + \sin t\, j + t k$

find $\vec{T}$ taking $\boxed{t_0 = 0}$ using $(*)$.

Sol. $s(t) = \int_0^t |V(\tau)| d\tau$

Now, $V(t) = \vec{r}\,'(t) = -\sin t\, i + \cos t\, j + k$

$|\vec{V}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\therefore s(t) = \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t$

$\Longrightarrow \boxed{t = \frac{s}{\sqrt{2}}}$

$\therefore \vec{r}(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right) i + \sin\left(\frac{s}{\sqrt{2}}\right) j + \frac{s}{\sqrt{2}} k$

$\therefore \vec{T} = \frac{d\vec{r}}{ds} = \frac{-1}{\sqrt{2}} \sin\left(\frac{s}{\sqrt{2}}\right) i + \frac{1}{\sqrt{2}} \cos\left(\frac{s}{\sqrt{2}}\right) j + \frac{1}{\sqrt{2}} k$

15. Arc length Find the length of the curve

$$\mathbf{r}(t) = \left(\overset{x}{\sqrt{2}t}\right)\mathbf{i} + \left(\overset{y}{\sqrt{2}t}\right)\mathbf{j} + \left(\overset{z}{1 - t^2}\right)\mathbf{k}$$

from $\underbrace{(\overset{x}{0}, \overset{y}{0}, \overset{z}{1})}_{t=0}$ to $\underbrace{(\sqrt{2}, \sqrt{2}, 0)}_{t=1}$. $0 \le t \le 1$

Sol. $\vec{V}(t) = \dfrac{d\vec{r}}{dt} = \sqrt{2}\,i + \sqrt{2}\,j - 2t\,k$

$$|\vec{V}(t)| = \sqrt{2 + 2 + 4t^2} = 2\sqrt{1+t^2}$$

$$\therefore \quad L = \int_0^1 |V(t)|\,dt$$

8.3 Trigonometric Substitution

$$= 2\int_0^1 \sqrt{1+t^2}\,dt$$

Firstly,

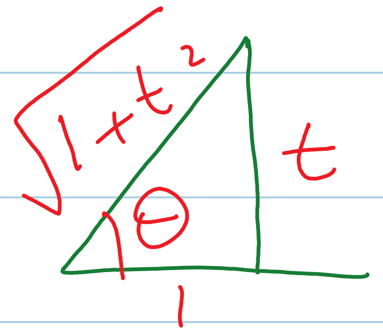
$$\cdot \int \sqrt{1+t^2}\,dt$$

$t = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$dt = \sec^2\theta\,d\theta$

$$= \int \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta$$

$$= \int \sqrt{\sec^2\theta}\,\sec^2\theta\,d\theta$$

$$= \int \sec^3\theta\,d\theta$$



by parts

$u = \sec\theta$ $dv = \sec^2\theta\,d\theta$

$du = \sec\theta\tan\theta\,d\theta$ $v = \tan\theta$

$$\therefore \int \sec^3\theta\,d\theta = uv - \int v\,du$$

$$= \sec\theta\tan\theta - \int \sec\theta\,\underbrace{(\tan^2\theta)}_{\sec^2\theta - 1}\,d\theta$$

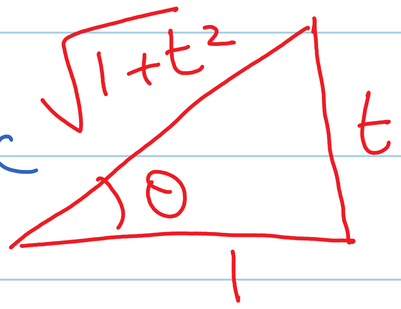
$$= \sec\theta\tan\theta - \int (\sec^3\theta - \sec\theta)\,d\theta$$

$$\therefore \int \sec^3\theta\, d\theta = \sec\theta \tan\theta + \int \sec\theta\, d\theta \underbrace{- \int \sec^3\theta\, d\theta}_{}$$

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + C$$

$$\therefore \int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{2}\sqrt{1+t^2}\cdot t + \frac{1}{2}\ln\left|\sqrt{1+t^2}+t\right| + C$$



$$\therefore L = 2\int_0^1 \sqrt{1+t^2}\, dt$$

$$= \left(t\sqrt{1+t^2} + \ln\left|\sqrt{1+t^2}+t\right|\right)\Big]_0^1$$

$$= \left[\sqrt{2} + \ln(\sqrt{2}+1)\right] - [0]$$

$$= \sqrt{2} + \ln(\sqrt{2}+1).$$

الانحناء $\vec{N}$

13.4 Curvature and Normal Vectors of a Curve

Curvature of a Plane Curve

$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$

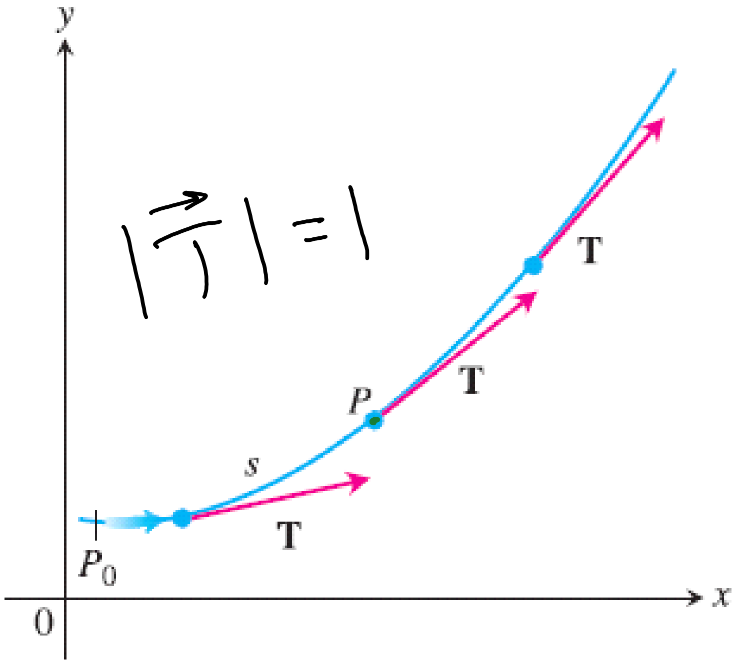


FIGURE 13.17 As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the *curvature* of the curve at P.

DEFINITION If **T** is the unit vector of a smooth curve, the **curvature** function of the curve is

$$\propto \quad \kappa = \left|\frac{d\mathbf{T}}{ds}\right|. \qquad K$$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$

$S = \int_0^t |v|.$

Rmk. If $|d\mathbf{T}/ds|$ is large, **T** turns sharply as the particle passes through P, and the curvature at P is large. If $|d\mathbf{T}/ds|$ is close to zero, **T** turns more slowly and the curvature at P is smaller.

Formula for Calculating Curvature

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|, \quad \checkmark$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Greek letter κ ("kappa").

$$K = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\right| \left|\frac{dt}{ds}\right| \rightarrow \frac{1}{|V|}$$

Ex. show that the curvature of a circle with radius a is $\kappa = \frac{1}{a} = \frac{1}{\text{radius}}$

Proof. $x^2+y^2=a^2$ $\qquad \kappa = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$

$x = a\cos t \quad , y = a\sin t$

$$\boxed{\vec{r}(t) = (a\cos t)i + (a\sin t)j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (-a\sin t)i + (a\cos t)j \checkmark$$

$$|\vec{v}| = \sqrt{(-a\sin t)^2+(a\cos t)^2} = \sqrt{a^2(1)}$$

$$= a \quad , a>0$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{a}(-a\sin t\, i + a\cos t\, j)$$

$$\boxed{\vec{T} = (-\sin t)i + (\cos t)j}$$

$$\frac{d\vec{T}}{dt} = (-\cos t)i - (\sin t)j$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

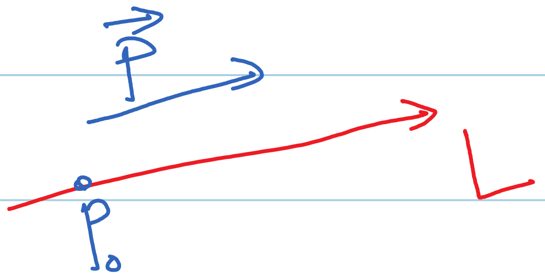
$$\therefore \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{a}\cdot 1 = \frac{1}{a}$$

$$= \frac{1}{\text{radius}}$$

ex. $x^2+y^2=9 \Rightarrow \kappa = \frac{1}{3}$

Ex. Show that the curvature of a line is zero.

constant



$\vec P \parallel L$

Sol. $\vec r(t) = P_0 + t\vec P$

$$\vec v = \frac{d\vec r}{dt} = \vec P$$

$$|\vec v| = |\vec P|$$

$$\therefore \vec T = \frac{\vec P}{|\vec P|}$$ constant vector

$$\frac{d\vec T}{dt} = \vec 0 \Rightarrow \kappa = \frac{1}{|\vec v|}\left|\frac{d\vec T}{dt}\right|$$

$$= \frac{1}{|\vec v|}|\vec 0| = 0.$$

DEFINITION At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\boxed{\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.}$$

$s = \int_0^t |v(\tau)|d\tau$

$\frac{ds}{dt} = |\vec v|$

$$\vec N = \frac{\frac{d\vec T}{ds}}{\left|\frac{d\vec T}{ds}\right|} = \frac{\frac{d\vec T}{dt}\cdot\cancel{\frac{dt}{ds}}}{\left|\frac{d\vec T}{dt}\right|\cancel{\left|\frac{dt}{ds}\right|}} \quad = \frac{dt}{ds} = \frac{1}{|\vec v|} > 0$$

$$\Rightarrow \boxed{\vec N = \frac{\frac{d\vec T}{dt}}{\left|\frac{d\vec T}{dt}\right|}}$$ ✓

Formula for Calculating N

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\boxed{\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},}$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

EXAMPLE 3 Find **T** ~~and~~ **N** and κ for the circular motion

$$\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j}.$$

Sol.

$$\vec{v} = \frac{d\vec{r}}{dt} = (-2\sin 2t)\,i + (2\cos 2t)\,j$$

$$|\vec{v}| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = 2$$

$$\vec{T} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{2}\left(-2\sin 2t\, i + 2\cos 2t\, j\right)$$

$$\boxed{\vec{T} = -\sin(2t)\, i + \cos(2t)\, j}$$

$$\frac{d\vec{T}}{dt} = (-2\cos 2t)\, i - (2\sin 2t)\, j$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$$

$$\vec{N} = \frac{1}{\left|\frac{d\vec{T}}{dt}\right|}\frac{d\vec{T}}{dt} = \frac{1}{2}\left[-2\cos 2t\, i - 2\sin 2t\, j\right]$$

$$= (-\cos 2t)\, i - (\sin 2t)\, j$$

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{2}(2) = 1.$$

$\kappa, \vec{T}$ and $\vec{N}$

(i)

EXAMPLE 5 Find the curvature for the helix (Figure 13.22)

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$

Sol:

$$\vec{V}(t) = \frac{d\vec{r}}{dt} = (-a\sin t)\,i + (a\cos t)j + bk$$

$$|\vec{V}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \sqrt{a^2+b^2}$$

$$\vec{T} = \frac{\vec{V}}{|\vec{V}|} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin t\, i + a\cos t j + bk\right)$$

$$\vec{T} = \frac{-a\sin t}{\sqrt{a^2+b^2}}\,i + \frac{a\cos t}{\sqrt{a^2+b^2}}\,j + \frac{b}{\sqrt{a^2+b^2}}\,k$$

$$\frac{d\vec{T}}{dt} = -\frac{a\cos t}{\sqrt{a^2+b^2}}\,i - \frac{a\sin t}{\sqrt{a^2+b^2}}\,j$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{a^2\cos^2 t}{a^2+b^2} + \frac{a^2\sin^2 t}{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\vec{N} = \frac{d\vec{T}}{dt} \Big/ \left|\frac{d\vec{T}}{dt}\right| = (-\cos t)\,i - (\sin t)\,j.$$

$$\kappa = \frac{1}{|\vec{V}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sqrt{a^2+b^2}}\,\frac{a}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$$

$$\kappa(a,b) = \frac{a}{a^2+b^2}$$

(ii) What's the largest value of K can have for a given value b ?

Sol. $K(a) = \frac{a}{a^2+b^2}$, b constant.

$$K'(a) = \frac{(a^2+b^2)(1) - a(2a)}{(a^2+b^2)^2} = \frac{b^2-a^2}{(a^2+b^2)^2}$$

$$K'(a) = 0 \Rightarrow b^2-a^2=0$$

$$a = \pm b$$

∴ The max. value of K occurs at a = b

$$\text{max. } K = K(b) = \frac{b}{b^2+b^2} = \frac{\cancel{b}}{2b^{\cancel{2}}}$$

$$= \frac{1}{2b}.$$

Circle of Curvature for Plane Curves

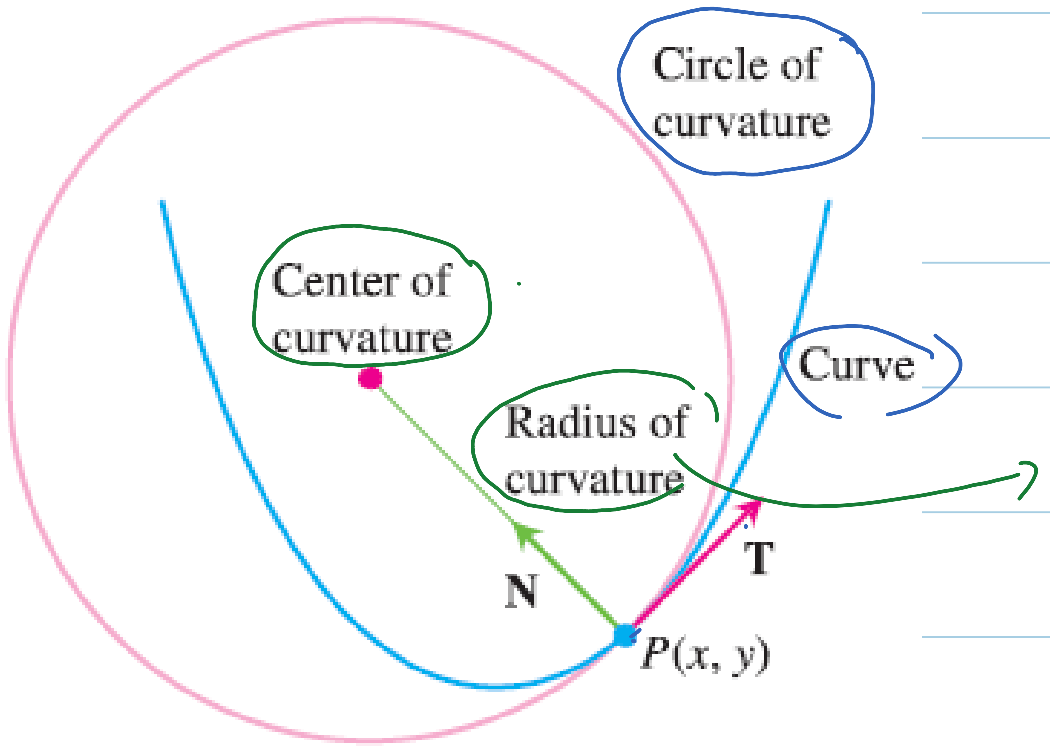
The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. lies toward the concave or inner side of the curve (as in Figure 13.20).

The **radius of curvature** of the curve at P is the radius of the circle of curvature, which, according to Example 2, is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

To find ρ, we find κ and take the reciprocal. The **center of curvature** of the curve at P is the center of the circle of curvature.



$\rho = \frac{1}{\kappa}, \kappa \neq 0$

FIGURE 13.20 The osculating circle at $P(x, y)$ lies toward the inner side of the curve.

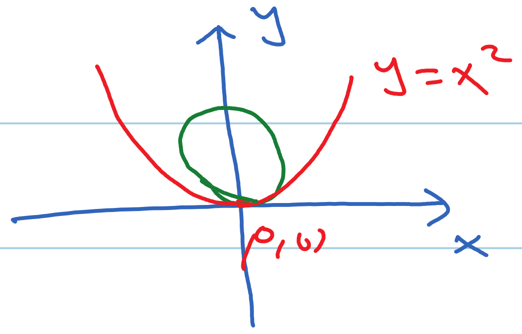
EXAMPLE 4 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Sol:

$y = x^2$

$x = t, \quad y = t^2$

$\vec{r}(t) = t\,i + t^2\,j$



$$\vec{r} = ti + t^2 j \,.$$

(t, t^2)
(x, y) $t=0$
$= (0,0)$

$$\vec{v} = \frac{d\vec{r}}{dt} = i + 2tj$$

$$|\vec{v}| = \sqrt{1 + 4t^2}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1+4t^2}}\, i + \frac{2t}{\sqrt{1+4t^2}}\, j$$

$$k(0) = \frac{1}{|\vec{v}(0)|} \left| \frac{d\vec{T}}{dt}(0) \right|$$

$$\vec{T} = (1+4t^2)^{-\frac{1}{2}} i + 2t(1+4t^2)^{-\frac{1}{2}} j$$

$$\frac{d\vec{T}}{dt} = \left[-\tfrac{1}{2}(1+4t^2)^{-3/2}(8t)\right] i + \left[2(1+4t^2)^{-\frac{1}{2}} + 2t(-\tfrac{1}{2})(1+4t^2)^{-3/2} . 8t\right] j$$

$$\frac{d\vec{T}}{dt}(0) = 2j \Rightarrow \left|\frac{d\vec{T}}{dt}(0)\right| = 2$$

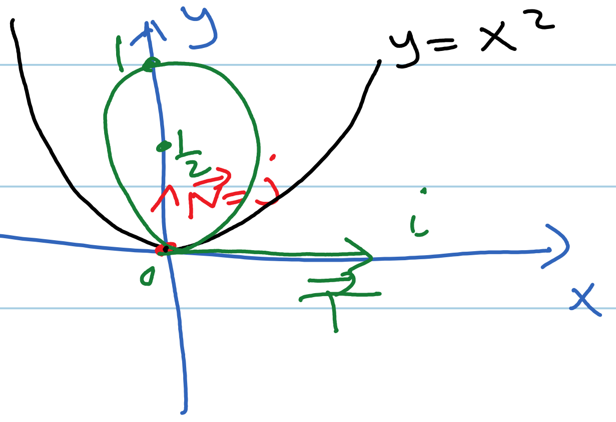
$$|\vec{v}(0)| = \sqrt{1 + 4(0)^2} = 1 \,.$$

$$\therefore \; k = \frac{1}{|\vec{v}(0)|} \left|\frac{d\vec{T}}{dt}(0)\right| = \frac{1}{1} . 2 = 2$$

$\therefore$ radius of curvature $= \rho = \dfrac{1}{k} = \dfrac{1}{2}$.

H.W

$$\boxed{\begin{aligned} \vec{T}(0) &= i \\ \vec{N}(0) &= j \end{aligned}}$$



radius $= \frac{1}{2}$

Center $(0, \frac{1}{2})$

$\therefore$ the eq. of the circle of curvature is

$$(x-0)^2+\left(y-\tfrac{1}{2}\right)^2=\left(\tfrac{1}{2}\right)^2$$

$$\boxed{x^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{1}{4}}$$

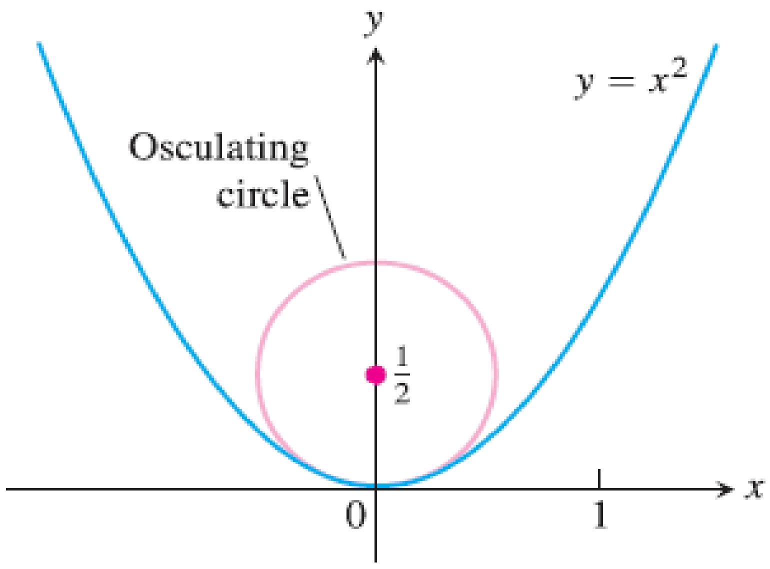


FIGURE 13.21 The osculating circle for the parabola $y = x^2$ at the origin (Example 4).

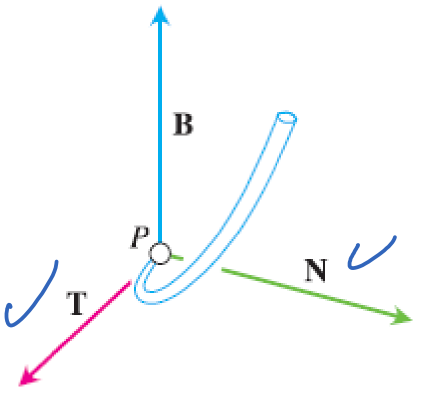
a_T a_N $\vec{a}(t) = \frac{d\vec{V}}{dt}$

13.5 Tangential and Normal Components of Acceleration

$\vec{T} = \frac{\vec{V}}{|\vec{V}|}$, $\vec{N} = \frac{\frac{dT}{dt}}{|\frac{dT}{dt}|}$

The TNB Frame

The **binormal vector** of a curve in space is **B** = **T** × **N**, a unit vector orthogonal to both **T** and **N** (Figure 13.24). Together **T**, **N**, and **B** define a moving right-handed vector frame that plays a significant role in calculating the paths of particles moving through space. It is called the **Frenet** ("fre-*nay*") **frame** (after Jean-Frédéric Frenet, 1816–1900), or the **TNB frame**.



$|\vec{T}| = 1$
$|\vec{N}| = 1$

FIGURE 13.24 The vectors **T**, **N**, and **B** (in that order) make a right-handed frame of mutually orthogonal unit vectors in space.

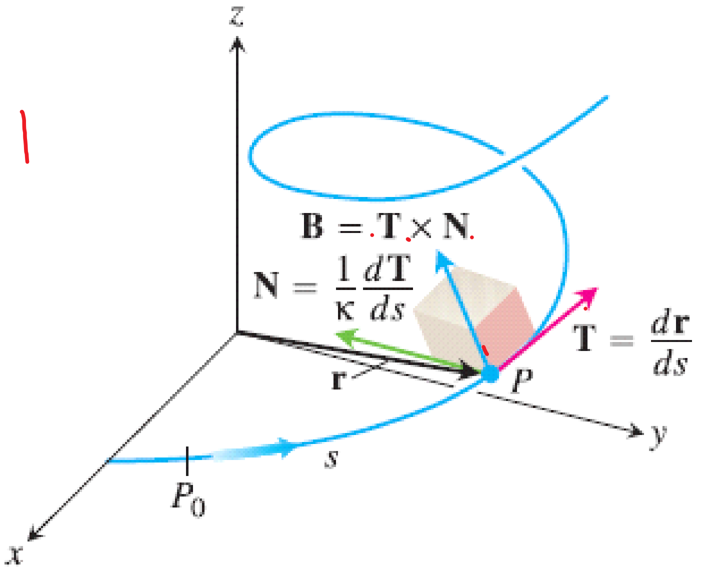
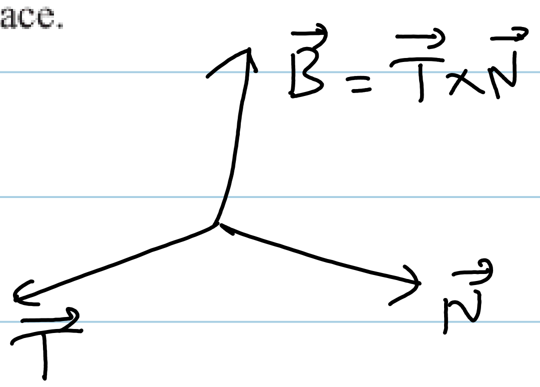


FIGURE 13.23 The **TNB** frame of mutually orthogonal unit vectors traveling along a curve in space.



Rmk. $\vec{B} \perp \vec{N}$ and $\vec{B} \perp \vec{T}$

$$|\vec{B}| = |\vec{T}||\vec{N}||\sin\theta|$$
$$= (1)(1)|\sin 90°| = 1$$

$\therefore$ $\vec{B}$ is unit vector.

Tangential and Normal Components of Acceleration

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\frac{\vec{V}}{|\vec{V}|}|\vec{V}|\right)$$
$$= \frac{d}{dt}\left(\vec{T} \cdot \frac{ds}{dt}\right) = \frac{dT}{dt}\frac{ds}{dt} + T\frac{d^2s}{dt^2}$$
$$= \frac{dT}{ds}\left(\frac{ds}{dt}\right)^2 + \left(\frac{d}{dt}|\vec{V}|\right)\vec{T}$$
$$= \frac{dT}{ds}|\vec{V}|^2 + \left(\frac{d}{dt}|\vec{V}|\right)\vec{T}$$

$$\vec{a} = \underbrace{\boxed{\kappa|V|^2}}_{a_N}\vec{N} + \underbrace{\left(\frac{d}{dt}|\vec{v}|\right)}_{a_T}\vec{T}$$

DEFINITION If the acceleration vector is written as

$$\boxed{\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N},}$$

then

$$\boxed{a_{\mathrm{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}|} \quad \text{and} \quad \boxed{a_{\mathrm{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2}$$

are the **tangential** and **normal** scalar components of acceleration.

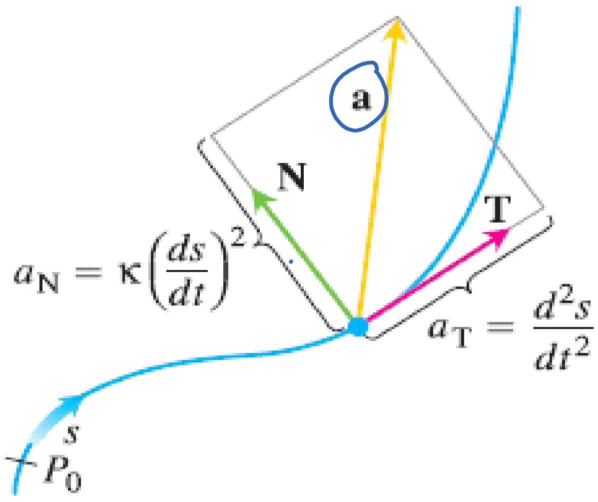


FIGURE 13.25 The tangential and normal components of acceleration. The acceleration **a** always lies in the plane of **T** and **N**, orthogonal to **B**.

Rmk: ① $\vec{a}$ always lies in the plane of $\vec{T}$ and $\vec{N}$ orthogonal to $\vec{B}$

② a_T measures the change in speed.

a_N measures the rate of change of the direction of $\vec{v}$.

Rmk: $|a|^2 = \vec{a}\cdot\vec{a} = (a_T\vec{T} + a_N\vec{N})\cdot(a_T\vec{T} + a_N\vec{N})$

$$= a_T^2\,\vec{T}\cdot\vec{T} + (2a_Ta_N)\cancelto{0}{\vec{T}\cdot\vec{N}} + a_N^2\,\vec{N}\cdot\vec{N}$$

$$= a_T^2\,\cancelto{1}{|\vec{T}|^2} + a_N^2\,\cancelto{1}{|\vec{N}|^2}$$

$$\therefore |\vec{a}|^2 = a_T^2 + a_N^2 \Rightarrow \boxed{a_N = \sqrt{|\vec{a}|^2 - a_T^2}}$$

EXAMPLE 1 Without finding **T** and **N**, write the acceleration of the motion

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \quad \boxed{t > 0}$$

in the form $\boxed{\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}.}$

Sol. $\vec{V} = \frac{d\vec{r}}{dt} = (-\cancel{\sin t} + \cancel{\sin t} + t\cos t)\, i$

$+ (\cancel{\cos t} - \cancel{\cos t} + t\sin t)\, j$

$$\vec{V} = (t\cos t)\, i + (t\sin t)\, j$$

$$|\vec{V}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2\,(1)} = |t| = t, \ t > 0$$

$$a_T = \frac{d}{dt}|V| = \frac{d}{dt}(t) = 1$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{V}}{dt}$$

$$= (\cos t - t\sin t)\, i + (\sin t + t\cos t)\, j$$

$$|\vec{a}|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2$$

$$= \boxed{\cos^2 t} - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \boxed{\sin^2 t}$$

$$+ \cancel{2t\sin t\cos t} + t^2\cos^2 t$$

$$|\vec{a}|^2 = 1 + t^2$$

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2} = \sqrt{(t^2+1) - (1)^2}$$

$$= \sqrt{t^2} = |t| = t$$

$$\vec{a} = a_T\vec{T} + a_N\vec{N} = 1\cdot\vec{T} + t\vec{N} = \vec{T} + t\vec{N}$$

Torsion (τ) ← tau

$$B = T \times N$$

تذكر

$$\frac{dB}{ds} = \frac{dT}{ds} \times N + T \times \frac{dN}{ds} \qquad \left\{ N = \frac{1}{\kappa}\frac{dT}{ds} \right\}$$

$$= \frac{dT}{ds} \times \left(\frac{1}{\kappa} \frac{dT}{ds} \right) + T \times \frac{dN}{ds}$$

$$= \frac{1}{\kappa}\left(\cancel{\frac{dT}{ds} \times \frac{dT}{ds}} \right)^{\vec{0}} + T \times \frac{dN}{ds}$$

$$\boxed{\frac{dB}{ds} = T \times \frac{dN}{ds}}$$

$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$
$|\vec{r}| = \text{Constant}$ ✓

Now, We know $B \perp \frac{dB}{ds}$ (Since $|\vec{B}| = 1$)

ثابت

$$\Rightarrow \frac{dB}{ds} \parallel N \Rightarrow \frac{dB}{ds} = -\tau N$$

$$\frac{dB}{ds} \cdot N = -\tau (N \cdot N) \qquad |N|^2 = 1$$

$$\therefore \tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$

$$= \left(-\frac{dB}{dt} \frac{dt}{ds} \right) \cdot \vec{N} \qquad \frac{dt}{ds} \to \frac{1}{|\vec{v}|}$$

$$\boxed{\tau = -\frac{1}{|\vec{v}|}\left(\frac{d\vec{B}}{dt} \cdot \vec{N} \right)}$$

torsion.

where $\vec{B} = \vec{T} \times \vec{N}$.

or $$\boxed{\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{V} \times \vec{a}|^2}} \quad , \vec{V} \times \vec{a} \neq \vec{0}$$

$$\vec{r} = xi + yj + zk$$

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}, \quad \dddot{x} = \frac{d^3x}{dt^3}.$$

. . . . .

Q8)

8. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad t = 0$

Find $\tau(0)$.

Sol.

$$\vec{V} = \vec{r}\,'(t) = (-\sin t)\,i + (\cos t)\,j + k$$

$$\vec{a} = \frac{d\vec{V}}{dt} = (-\cos t)\,i - \sin t\; j$$

$$\vec{V}(0) = j + k, \quad \vec{a}(0) = -i$$

$$\vec{V}(0) \times \vec{a}(0) = \begin{vmatrix} \overset{\oplus}{i} & \overset{\ominus}{j} & \overset{\oplus}{k} \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= 0i - (0+1)j + (0+1)k$$

$$= -j + k$$

$$|\vec{V}(0) \times \vec{a}(0)| = \sqrt{1+1} = \sqrt{2}.$$

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \\ \sin t & -\cos t & 0 \end{vmatrix} \overset{\text{at } t=0}{=} \begin{vmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= 1$$

$$\therefore \tau(0) = \frac{\begin{vmatrix} \dot{x}(0) & \dot{y}(0) & \dot{z}(0) \\ \ddot{x}(0) & \ddot{y}(0) & \ddot{z}(0) \\ \dddot{x}(0) & \dddot{y}(0) & \dddot{z}(0) \end{vmatrix}}{\left|\vec{V}(0) \times \vec{a}(0)\right|^2}$$

$$= \frac{1}{(\sqrt{2})^2} = \frac{1}{2} .$$

Summary

Computation Formulas for Curves in Space

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ ✓

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ ✓

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ ✓

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \dfrac{1}{|\vec{V}|}\left|\dfrac{d\vec{T}}{dt}\right|$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = -\dfrac{1}{|\vec{V}|}\left(\dfrac{d\vec{B}}{dt} \cdot \vec{N}\right)$

Tangential and normal scalar components of acceleration: $\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$

$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}|$$

$$a_{\mathrm{N}} = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}^2}$$

12.5

$P(x_0, y_0, z_0)$

$\vec{n} = Ai + Bj + ck$

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

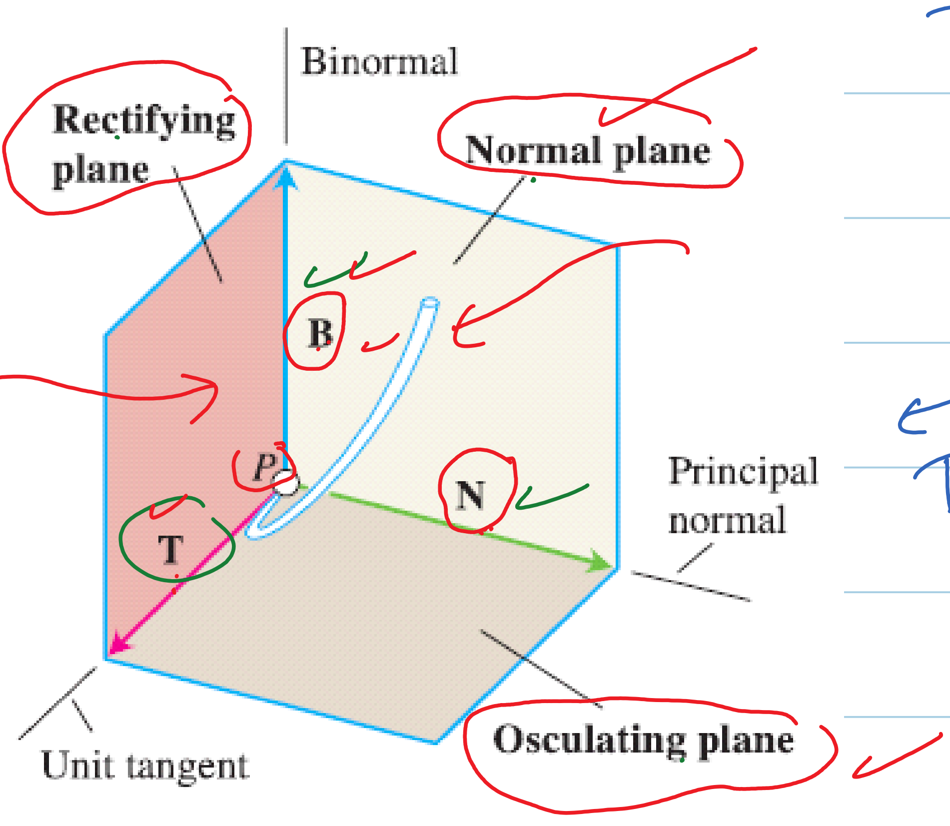
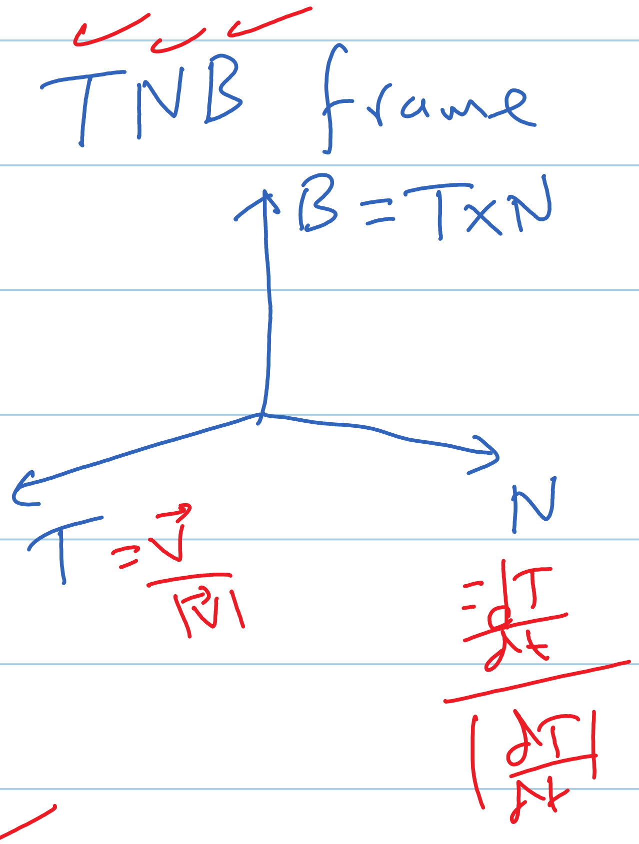


FIGURE 13.28 The names of the three planes determined by **T**, **N**, and **B**.



$B = T \times N$

$T = \frac{\vec{v}}{|\vec{v}|}$

$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$

13.5 + 12.5

In Exercises 7 and 8, find **r**, **T**, **N**, and **B** at the given value of t. Then find equations for the osculating, normal, and rectifying planes at that value of t.

8. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad t = 0$

$P \to ??$ At $t = 0$, $\vec{r}(0) = \cos 0\, i + \sin 0\, j + 0k$

$= i$

$\therefore P(1, 0, 0).$

$\vec{v} = \frac{d\vec{r}}{dt} = (-\sin t)i + (\cos t)j + k$

$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{-\sin t}{\sqrt{2}} i + \frac{\cos t}{\sqrt{2}} j + \frac{1}{\sqrt{2}} k$

$t = 0, \quad \vec{T}(0) = \quad \frac{1}{\sqrt{2}} j + \frac{1}{\sqrt{2}} k \qquad P(1, 0, 0)$

Normal plane

$0(x - 1) + \frac{1}{\sqrt{2}}(y - 0) + \frac{1}{\sqrt{2}}(z - 0) = 0$

$\Rightarrow \quad y + z = 0$

$$\vec{N} = \frac{d\vec{T}}{dt} \Big/ \left|\frac{d\vec{T}}{dt}\right| .$$

$$\vec{T} = \frac{1}{\sqrt{2}}(-\sin t\, i + \cos t\, j + k)$$

$$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{2}}(-\cos t\, i - \sin t\, j)$$

$|\alpha\vec{V}|$
$= |\alpha||\vec{V}|$

$$\left|\frac{d\vec{T}}{dt}\right| = \left|\frac{1}{\sqrt{2}}\right| \sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}$$

$$\vec{N} = \frac{d\vec{T}}{dt} \Big| \left|\frac{d\vec{T}}{dt}\right| = -\cos t\, i - \sin t\, j$$

$\vec{N}(0) = -i$, $P(1,0,0)$

Rectifying plane $-1(x-1)+0(y-0)+0(z-0)=0$

$\Rightarrow \boxed{x=1}$

$$\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix}$$

$$\boxed{\vec{B}(0) = -\frac{1}{\sqrt{2}} j + \frac{1}{\sqrt{2}} k}$$

osculating plane $P(1,0,0)$

$$0(x-1) - \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0$$

or $\boxed{y = z}$

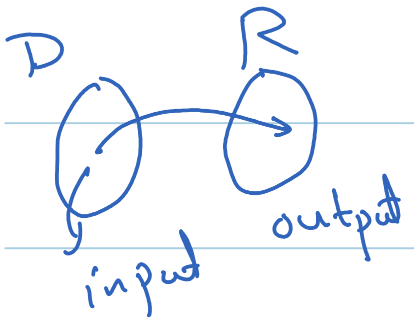
the End of ch13.

14 مشتقات جزئية

PARTIAL DERIVATIVES

14.1 Functions of Several Variables

Cal 1 $y = f(x)$ dep. indep



DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

$\sqrt{x}$

$y = f(x)$

Remarks. ① If f is a function of two independent variables $\boxed{z = f(x,y)}$, then we picture the domain of f as a region in the xy-plane.

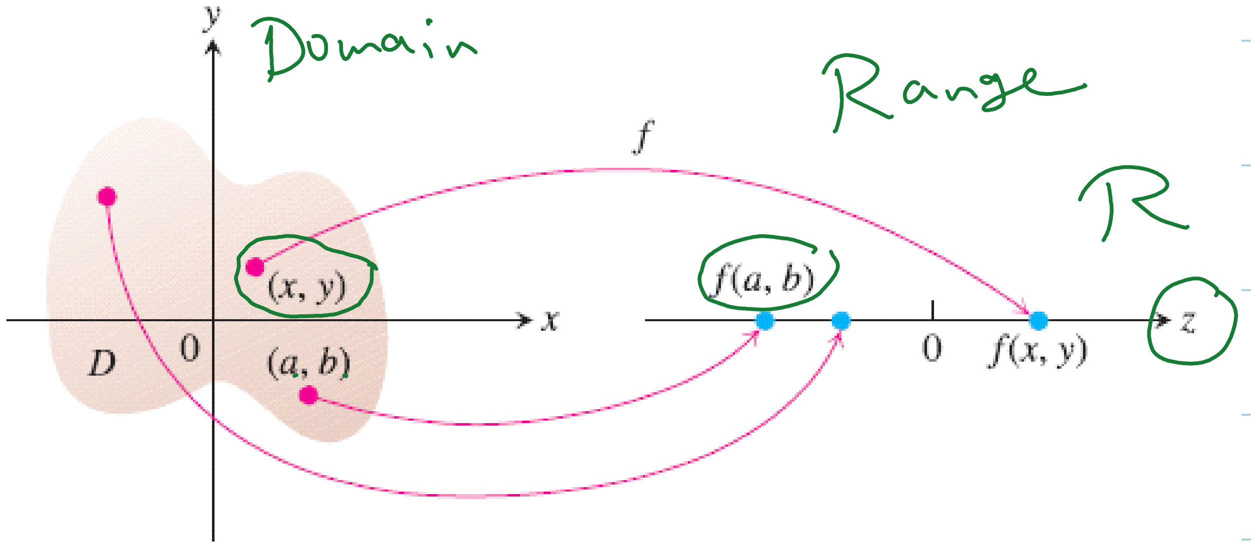


FIGURE 14.1 An arrow diagram for the function $z = f(x, y)$.

② If f is a function of three variables $\boxed{w = f(x,y,z)}$, we picture the domain of f as a region in Space.

Ex. If $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Find $f(\overset{x}{3}, \overset{y}{0}, \overset{z}{4})$

Sol. $f(3,0,4) = \sqrt{3^2+0^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$.

input output

Domains and Ranges

Example. ⓐ Find and sketch the function's domain

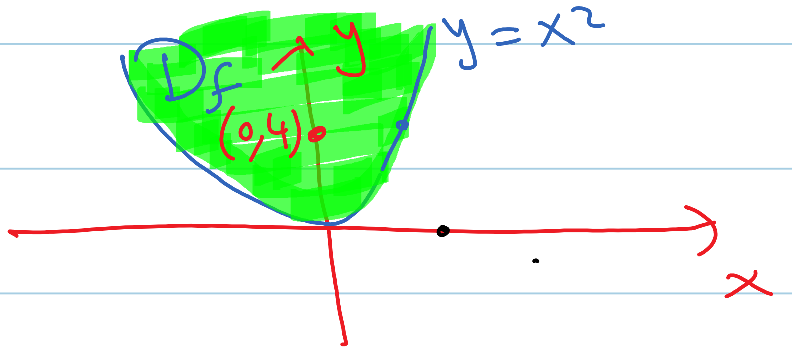
ⓑ Find the function's range.

① $z = f(x,y) = \boxed{\sqrt{y - x^2}}$

$D_f = \{(x,y) : y - x^2 \geq 0\} = \{(x,y) : y \geq x^2\}$

unbounded

closed



$4 \geq 0^2$ ✓

$R_f = [0, \infty)$

$y - x^2 \geq 0$

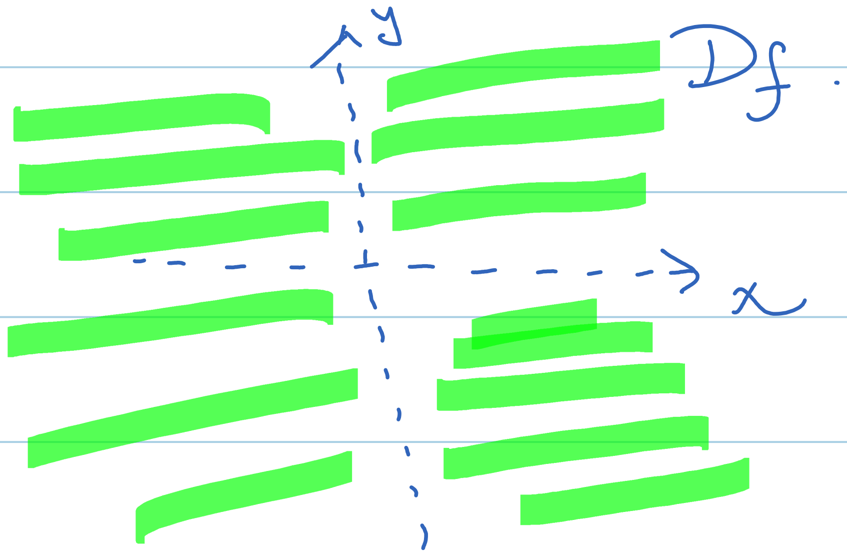
$z = \sqrt{y - x^2} \geq 0 \Rightarrow \boxed{z \geq 0}$

② $z = f(x,y) = \frac{1}{xy} = 2$

$D_f = \{(x,y) : x \neq 0 \text{ and } y \neq 0\}$.

unbounded

open.



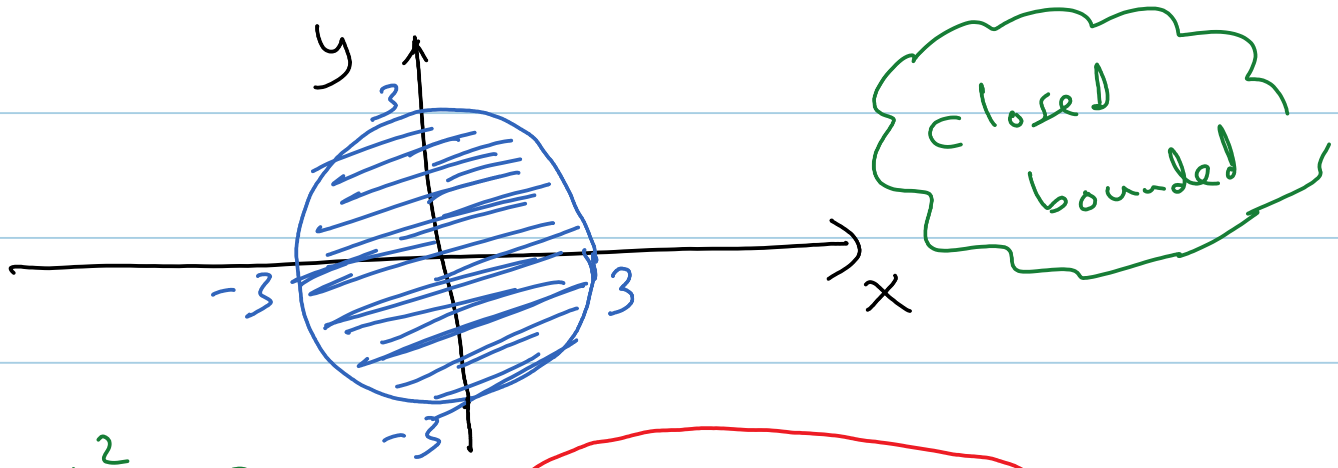
$R_f = (-\infty, 0) \cup (0, \infty)$

$= \mathbb{R} \setminus \{0\}$.

③ $z = f(x,y) = \sqrt{9-x^2-y^2}$

$$D_f = \{(x,y) : 9-x^2-y^2 \geq 0\}$$

$$= \{(x,y) : x^2+y^2 \leq 9\}$$



$R_f : 0 \leq x^2+y^2 \leq 9$ ✓

$\boxed{z = \sqrt{9-x^2-y^2}}$

$$\underset{+9}{-9} \leq \underset{+9}{-x^2-y^2} \leq \underset{+9}{0}$$

$$0 \leq 9-x^2-y^2 \leq 9$$

$$0 \leq \underbrace{\boxed{\sqrt{9-x^2-y^2}}}_{z} \leq 3$$

$$0 \leq z \leq 3$$

$\therefore$ $R_f = [0,3]$.

$-1 \le x \le 1$

④ $f(x,y) = 4\sin^{-1}(y-2x)$.

$D_{\sin^{-1}x} : -1 \le x \le 1$

$R_{\sin^{-1}x} : -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$D_f = \{(x,y) : -1 \le y-2x \le 1\}$$

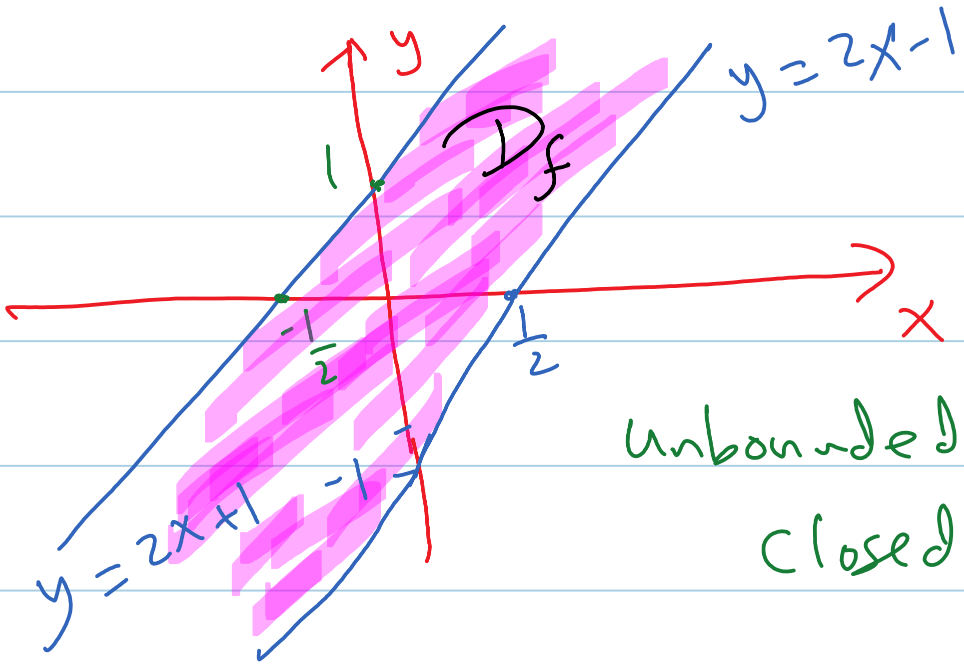
$$= \{(x,y) : 2x-1 \le y \le 2x+1\}$$

$y = 2x-1$

$x = 0 \Rightarrow y = -1$

$y = 0 \Rightarrow 2x-1 = 0$

$x = \frac{1}{2}$



Unbounded

Closed

Range:

$$-\frac{\pi}{2} \le \sin^{-1}(y-2x) \le \frac{\pi}{2}$$

$$-4\frac{\pi}{2} \le \underbrace{4\sin^{-1}(y-2x)}_{z} \le 4\frac{\pi}{2}$$

$$-2\pi \le z \le 2\pi.$$

$$\therefore R_f = [-2\pi, 2\pi].$$

⑤ $z = f(x,y) = e^{-x^2-y^2} = \frac{1}{e^{x^2+y^2}} > 0$

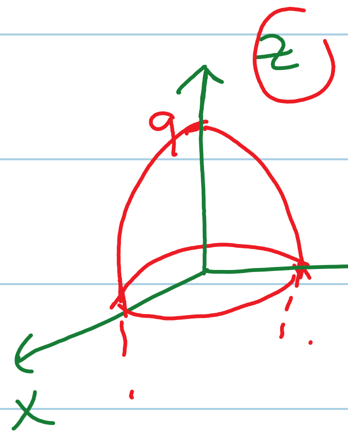
D_f = Entire plane (clopen (closed and open)

$R_f = (0,1]$ $x^2+y^2 \geq 0 \Rightarrow 0 < e^{-x^2-y^2} \le e^0 = 1$

$0 < z \le 1$

⑥ $z = f(x,y) = 9 - x^2 - y^2 = 9 - (x^2+y^2)$

D_f = Entire plane.



R_f : $z = 9 - (\underbrace{x^2+y^2}_{\geq 0}) \leq 9$

$x^2+y^2 \geq 0 \Rightarrow -x^2-y^2 \leq 0$

$9-x^2-y^2 \leq 9$

$-\infty < z \leq 9$

$\therefore R_f = (-\infty, 9]$

⑦ $w = f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$D_f = \{(x,y,z) : x^2+y^2+z^2 \geq 0\}$

= Entire space.

$R_f = [0, \infty)$.

⑧ $w = f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

$D_f = \{(x,y,z) : (x,y,z) \neq (0,0,0)\}$

= Entire space $\setminus \{(0,0,0)\}$

$R_f = (0, \infty)$.

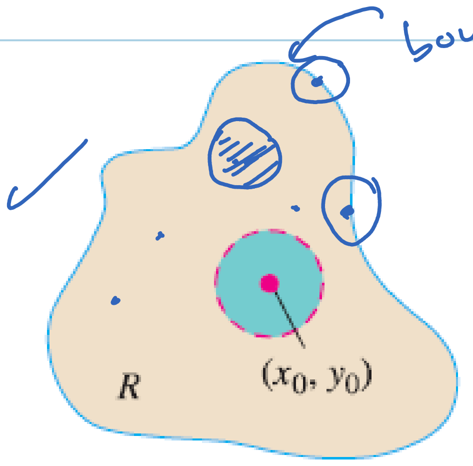
⑨ $w = xy\ln z$, $R_w = (-\infty, \infty)$

$D_w = \{(x,y,z) : z > 0\}$ = Half-space.

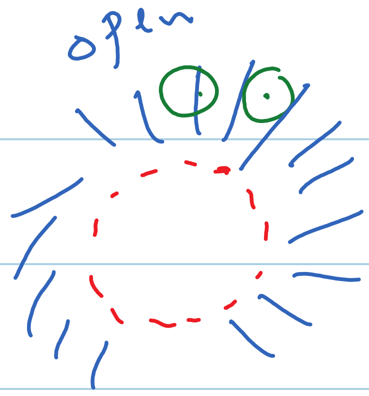
نقاط داخلية

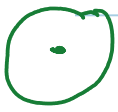
DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

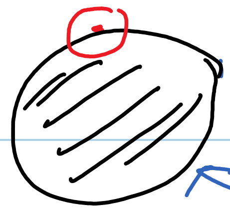


(a) Interior point

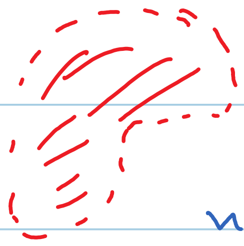




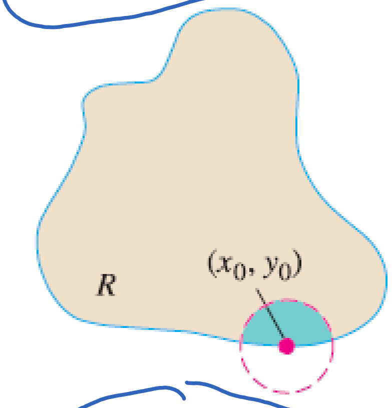
exterior point
خارجية



not open
closed

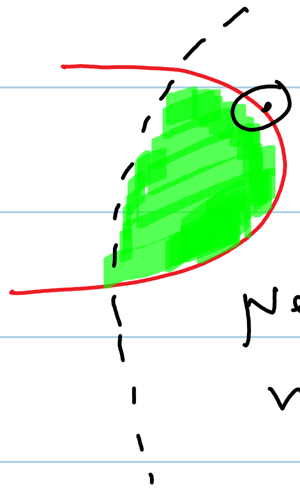


open
not closed.

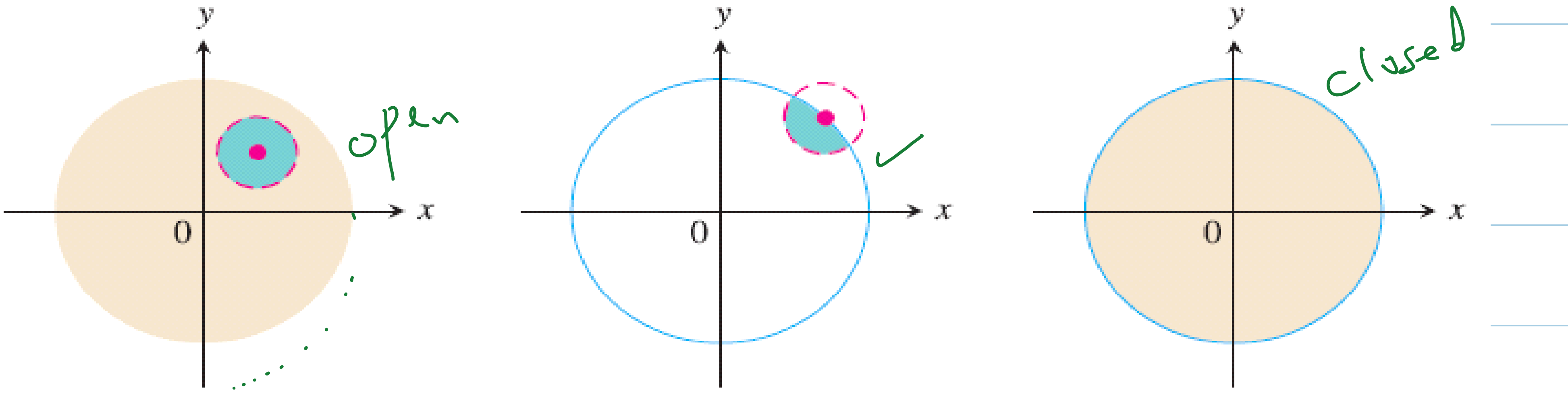


(b) Boundary point

Entire plane
closed & open
clopen



not open
not closed
Neither closed
nor open.



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an
interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit
disk. (The unit
circle.)

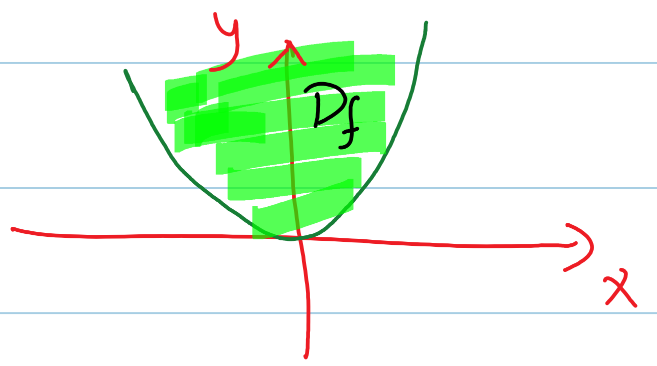
$\{(x, y) \mid x^2 + y^2 \le 1\}$
Closed unit disk.
Contains all
boundary points.

FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Ex: $f(x,y) = \sqrt{y-x^2}$ find and decribe D_f.

$D_f = \{(x,y) : y \geq x^2\}$



closed, unbounded.

Interior points = $\{(x,y) : y > x^2\}$.

Boundary points = $\{(x,y) : y = x^2\} = \{(x,x^2) : x \in \mathbb{R}\}$

Graphs, Level Curves, and Contours of Functions of Two Variables

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(\check{x}, \check{y}, \overset{z}{f(x, y)})$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the $\boxed{\textbf{surface } z = f(x, y)}$.

ex: Describe the level curve of

$z = f(x,y) = 100 - x^2 - y^2$

Sol. $f(x,y) = c$ " constant "

$100 - x^2 - y^2 = c$

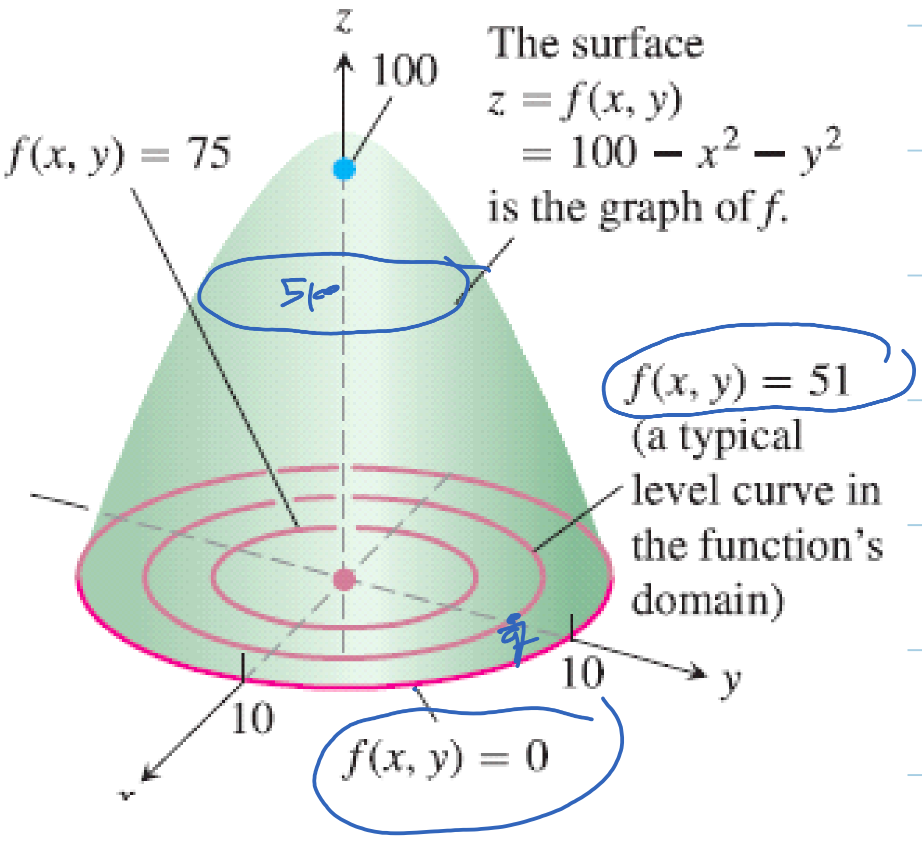
$$\boxed{x^2 \overset{+ve}{+} y^2 = 100 - c}$$

$c = 100 \implies x^2 + y^2 = 0 \implies (x,y) = (0,0)$ point

$100 - c > 0$ (i.e; $c < 100$)

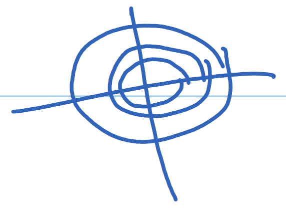
level curves are circles.

$c > 100$ No graph.



The surface $z = f(x, y) = 100 - x^2 - y^2$ is the graph of f.

$f(x, y) = 51$ (a typical level curve in the function's domain)



$f(x,y) = 51$

$\boxed{100 - x^2 - y^2 = 51}$

$\boxed{x^2 + y^2 = 49}$ level curve

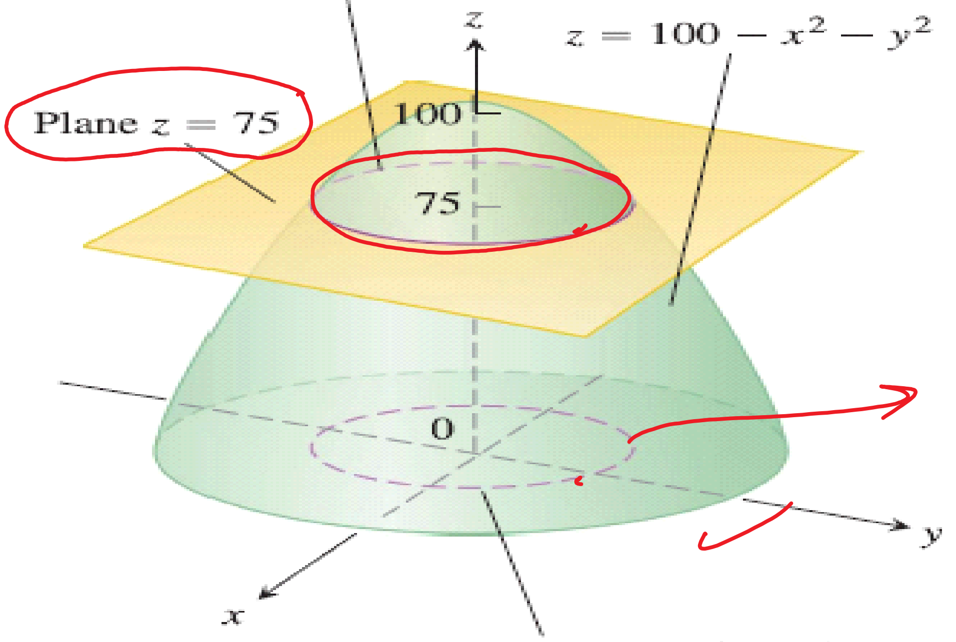
$100 - x^2 - y^2 = 0$

$x^2 + y^2 = 100$

contour: $\boxed{z = 100 - x^2 - y^2 \quad , \quad z = 51}$

$\boxed{x^2 + y^2 = 49}$

The contour curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the plane $z = 75$.



Level curve $100 - x^2 - y^2 = 75$

The level curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the xy-plane.

FIGURE 14.6 A plane $z = c$ parallel to the xy-plane intersecting a surface $z = f(x, y)$ produces a contour curve.

ex. the contour of

$$z = f(x,y) = 100 - x^2 - y^2 \quad , \quad z = 75$$

Sol. $100 - x^2 - y^2 = 75 \Rightarrow x^2 + y^2 = 25$

$\therefore$ contour is a circle in the plane $z = 75$.

Ex. Find an eq. of the level curve of $f(x,y) = 4\ln(3 - 2x^2 - y^2)$ passing through $(1,0)$.

Sol.

$$f(x,y) = c \Rightarrow f(x,y) = f(1,0)$$
$$f(1,0) = c$$

$$\Rightarrow 4\ln(3 - 2x^2 - y^2) = \cancelto{0}{4\ln(3-2)}$$

$$\ln(3 - 2x^2 - y^2) = 0$$

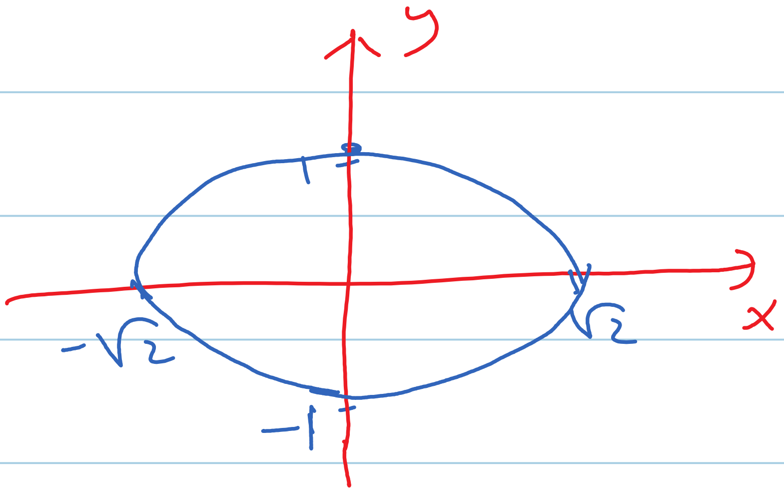
$$3 - 2x^2 - y^2 = e^0 = 1$$

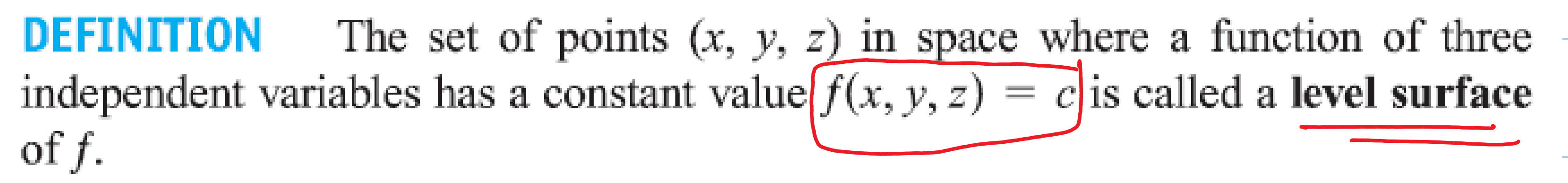
$$\Rightarrow \boxed{2x^2 + y^2 = 2} \Rightarrow x = 0 \Rightarrow y = \pm\sqrt{2}$$

$$y = 0 \Rightarrow x = \pm 1$$

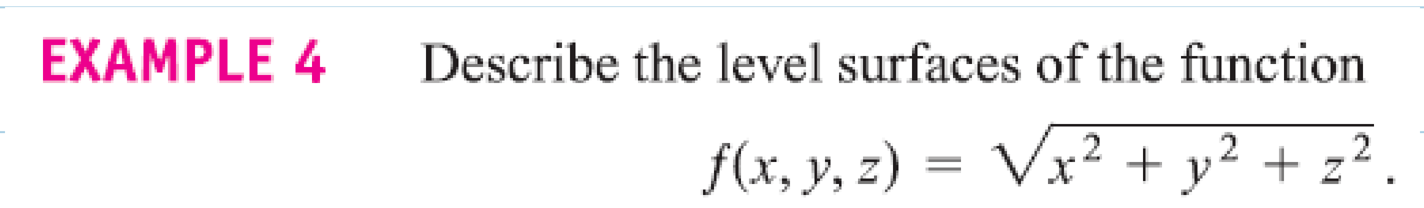
Sketch and identify.

ellipse





DEFINITION The set of points (x, y, z) in space where a function of three independent variables has a constant value $\boxed{f(x, y, z) = c}$ is called a **level surface** of f.



EXAMPLE 4 Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Sol.

$$f(x,y,z) = c$$

موجب

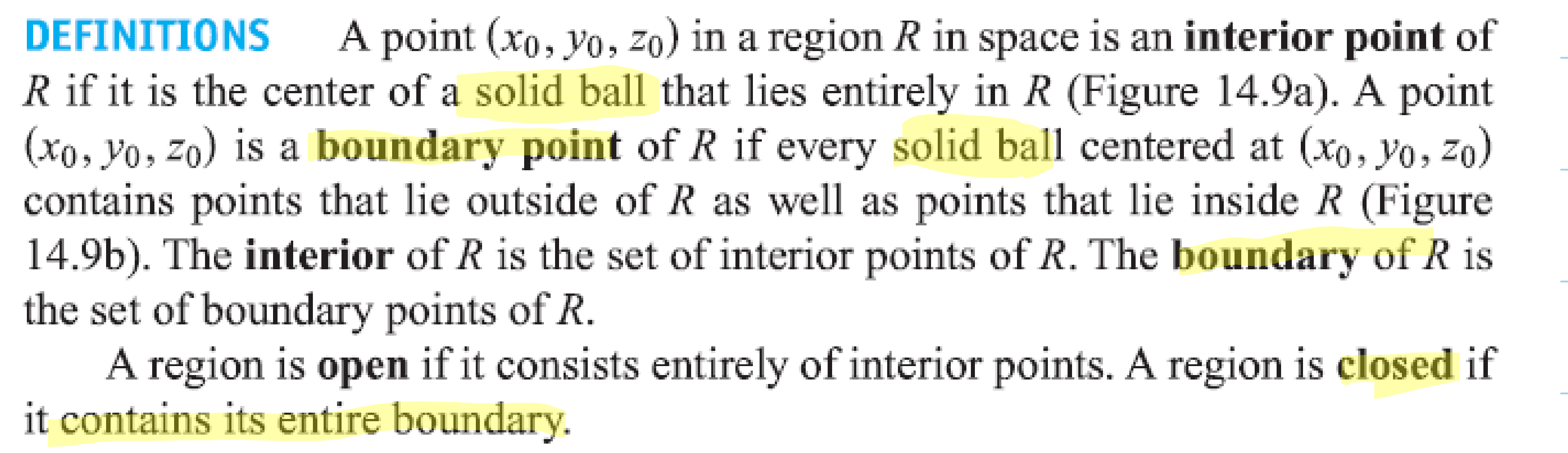
$$\sqrt{x^2+y^2+z^2} = c, \quad c>0$$

$$x^2+y^2+z^2 = c^2$$

the level surfaces are spheres if $c>0$

" " " " the origin if $c=0$
$(0,0,0)$

There is no graph if $c<0$.



DEFINITIONS A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if it is the center of a solid ball that lies entirely in R (Figure 14.9a). A point (x_0, y_0, z_0) is a **boundary point** of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie inside R (Figure 14.9b). The **interior** of R is the set of interior points of R. The **boundary** of R is the set of boundary points of R.

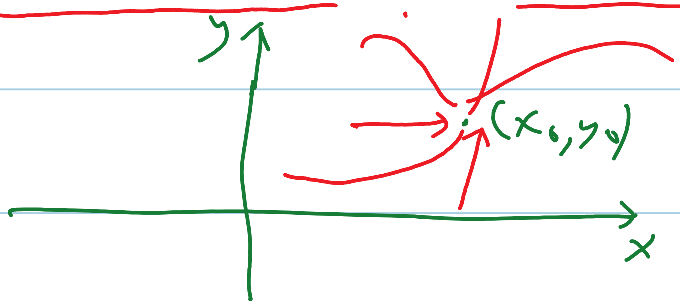
A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.

14.2 Limits and Continuity in Higher Dimensions

Limits for Functions of Two Variables

If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0), and write

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L}$$



THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) + g(x, y)) = L + M$ ✓
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) - g(x, y)) = L - M$ ✓
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x, y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x, y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

EXAMPLE 1

f(x,y)

(a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - 0(1) + 3}{0^2(1) + 5(0)(1) - 1^3} = \frac{3}{-1} = -3$$

exists

(b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$ exists

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EXAMPLE 2 Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}. \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0(0+0)$$

$$= 0$$

exists.

Ex③.

$$\lim_{(x,y)\to(2,-4)} \frac{y+4}{\underbrace{x^2y-xy}\,+\underbrace{4x^2-4x}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{(x,y)\to(2,-4)} \frac{y+4}{xy(x-1)+4x(x-1)}$$

$$= \lim_{(x,y)\to(2,-4)} \frac{\cancel{y+4}}{x(x-1)(\cancel{y}+4)} = \frac{1}{2(2-1)} = \frac{1}{2}$$

Ex4. $\lim\limits_{(x,y)\to(0,0)} y^2 \sin\left(\frac{1}{x}\right)$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 , \quad x \ne 0$$

$$-y^2 \le \underbrace{\boxed{y^2 \sin\left(\frac{1}{x}\right)}}_{f(x,y)} \le y^2 , \quad x \ne 0$$

Since $\lim\limits_{(x,y)\to(0,0)} -y^2 = 0$ and $\lim\limits_{(x,y)\to(0,0)} y^2 = 0$

then by Squeeze thm $\lim\limits_{(x,y)\to(0,0)} \left(y^2 \sin\frac{1}{x}\right) = 0$

Ex5 $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} \quad \left(\frac{0}{0}\right)$

$$u = x^2+y^2 , \quad (x,y)\to(0,0) \Rightarrow u \to 0$$

$$= \lim_{u\to 0} \frac{\sin(u)}{u} \overset{\text{L'Hôpital}}{=} \lim_{u\to 0} \frac{\cos(u)}{1} = 1$$

Ex.6 $\lim\limits_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2} \quad \left(\frac{0}{0}\right)$

Polar coordinates $\quad x = r\cos\theta$
$\qquad\qquad\qquad\qquad y = r\sin\theta$

$$x^2+y^2 = r^2 , \quad (x,y)\to(0,0) \Rightarrow r \to 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \overbrace{\frac{x^3-xy^2}{x^2+y^2}}^{f(x,y)} = \lim_{r\to 0} \overbrace{\frac{r^3\cos^3\theta - r\cos\theta . r^2\sin^2\theta}{r^2}}^{g(r,\theta)}$$

$$= \lim_{r\to 0} r\left(\cos^3\theta - \sin^2\theta\cos\theta\right)$$

$$= 0\,\overbrace{\left(\cos^3\theta-\sin^2\theta\cos\theta\right)}^{bdd}$$

$$= 0 \text{ exists.}$$

Q67

Ex 7 $\displaystyle \lim_{(x,y)\to(0,0)} \operatorname{Ln}\left(\frac{3x^2+3y^2-x^2y^2}{x^2+y^2}\right)$

$$= \operatorname{Ln}\left(\lim_{r\to 0} \frac{3r^2 - r^4\cos^2\theta\sin^2\theta}{r^2}\right)$$

$$= \operatorname{Ln}\left(\lim_{r\to 0}\left(3 - r^2\underbrace{\cos^2\theta\sin^2\theta}\right)\right)$$

$$= \operatorname{Ln}(3 - 0) = \operatorname{Ln} 3 \text{ exists}$$

Q64

Ex.8 $\displaystyle \lim_{(x,y)\to(0,0)} \frac{2x}{x^2+y^2+x}$

$r^2 = x^2+y^2$
$x = r\cos\theta$
$y = r\sin\theta$
$(x,y)\to(0,0) \Rightarrow r\to 0$

$$= \lim_{r\to 0}\left(\frac{2r\cos\theta}{r^2+r\cos\theta}\right)$$

$$= \lim_{r\to 0}\frac{2\cos\theta}{r+\cos\theta} = 2 \text{ if } \cos\theta \neq 0$$

limit DNE for $\cos\theta = 0$.

$\therefore$

Ex9. $\lim_{(x,y)\to(0,0)} \frac{4x^2y^2}{x^4+y^4}$

$$= \lim_{r\to 0} \frac{4r^2\cos^2\theta \cdot r^2\sin^2\theta}{r^4\cos^4\theta + r^4\sin^4\theta}$$

$$= \lim_{r\to 0} \frac{4\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}$$

$$= \frac{4\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta} \quad \text{DNE}$$

Since the limit varies for θ varies.

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Two-Path Test for Nonexistence of a Limit

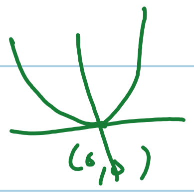
If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

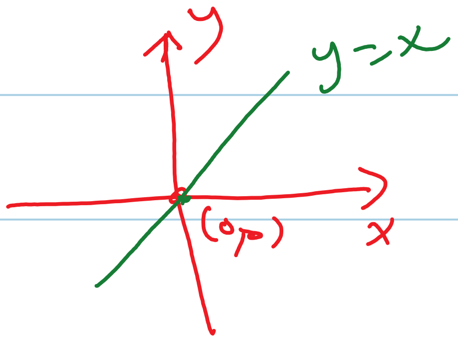
EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.14) has no limit as (x, y) approaches $(0, 0)$.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$





Sol. $\boxed{\text{Along } y = x}$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^2\cdot x}{x^4+x^2}$$

$$= \lim_{x\to 0} \frac{2x^3}{x^4+x^2} = \lim_{x\to 0} \frac{2x}{x^2+1} = \frac{0}{1} = \boxed{0}$$

$\boxed{\text{Along } y = x^2}$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{2x^2\cdot x^2}{x^4+(x^2)^2}$

$$= \lim_{x\to 0} 1 = \boxed{1}$$

By the Two-path test, $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ DNE

Ex. $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^3+3y^3} = \lim_{x\to 0} \frac{2x\cdot k^2x^2}{x^3+3k^3x^3}$

Along $\boxed{y = kx}$, $x \neq 0$ $= \lim_{x\to 0} \frac{2k^2}{1+3k^3}$

$$= \frac{2k^2}{1+3k^3}$$

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y=0

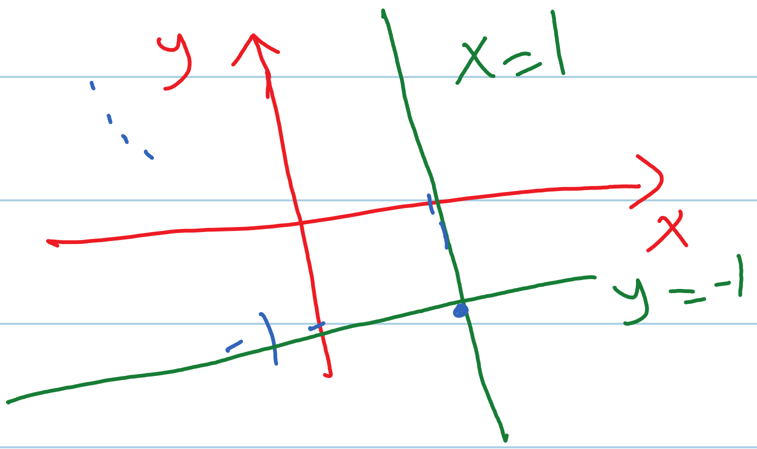
If $\{k=0\}$, $\lim_{(x,y)\to(0,0)} f(x,y) = \boxed{0}$

y=x

$\{k=1\}$, $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{2}{4} = \boxed{\frac{1}{2}}$

∴ $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE by Two-path test.

Ex. $\lim_{(x,y)\to(1,-1)} \left(\frac{xy+1}{x^2-y^2}\right)$



Along x=1,

$\lim_{y\to-1} \frac{y+1}{1-y^2}$ $\left(\frac{0}{0}\right)$

(a,b) x=a
y=b

$= \lim_{y\to-1} \frac{1}{-2y} = \frac{1}{-2(-1)} = \boxed{\frac{1}{2}}$

Along y=-1, $\lim_{x\to1} \frac{-x+1}{x^2-1} = \lim_{x\to1} \frac{-1}{2x} = \boxed{-\frac{1}{2}}$

∴ $\lim_{(x,y)\to(1,-1)} f(x,y)$ DNE by two-path test.

Ex. $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1} = \lim_{y\to1} \frac{y^2-1}{y-1} = \lim_{y\to1} \frac{2y}{1} = \boxed{2}$

Along x=1

Along y=x, $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1} = \lim_{x\to1} \frac{x^3-1}{x-1}$ $\frac{0}{0}$

$= \lim_{x\to1} 3x^2 = \boxed{3}$

∴ Two-path test ⇒ $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$ DNE.

Continuity

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. ✓ $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

EXAMPLE 5 Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.13).

① $f(0,0) = 0$ defined

② $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{x\to0} \frac{2x\cdot x}{x^2+x^2} = \boxed{1}$

Along $y = x$

$\boxed{\text{Along } y = x^2}$, $\lim_{x\to0} \frac{2x\cdot x^2}{x^2+x^4} = \lim_{x\to0} \frac{2x}{1+x^2} = \boxed{0}$

∴ $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ DNE by Two-Path test.

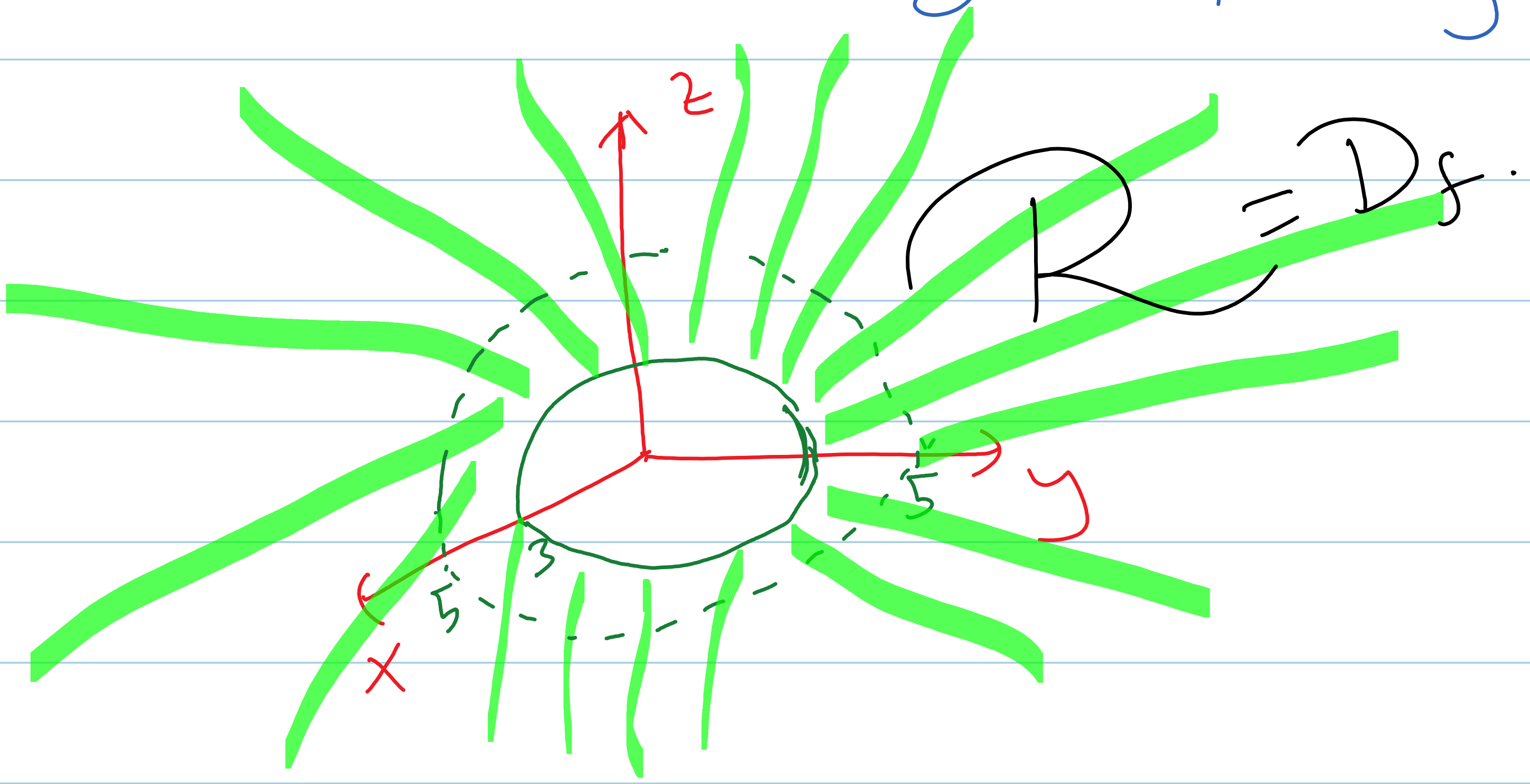
⇒ $f(x,y) = \frac{2xy}{x^2+y^2}$ is discont. at $(0,0)$

Ex. where $f(x,y,z) = \dfrac{1}{4-\sqrt{x^2+y^2+z^2-9}}$

is continuous ?.

Sol. $\text{Domain}(f) = \{(x,y,z) : x^2+y^2+z^2-9 \geq 0$ and $4-\sqrt{x^2+y^2+z^2-9} \neq 0\}$

$= \{(x,y,z) : x^2+y^2+z^2 \geq 9$ and $x^2+y^2+z^2 \neq 25\}$



$\therefore f$ is cont. on $R = D_f$.

مشتقات جزئية

$y = f(x)$

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

14.3 Partial Derivatives

Partial Derivatives of a Function of Two Variables

$Z = f(x, y)$

DEFINITION The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, \boxed{y_0}) - f(x_0, \boxed{y_0})}{h},$$

provided the limit exists.

f_x " f sub x " f_y " f sub y "

$= \frac{\partial f}{\partial y}$

We use several notations for the partial derivative:

$$\checkmark \ \frac{\partial f}{\partial x}(x_0, y_0) \text{ or } \boxed{f_x(x_0, y_0),} \qquad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \qquad \text{and} \qquad f_x, \ \frac{\partial f}{\partial x}, \ z_x, \text{ or } \frac{\partial z}{\partial x}.$$

f sub x

DEFINITION The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y = y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

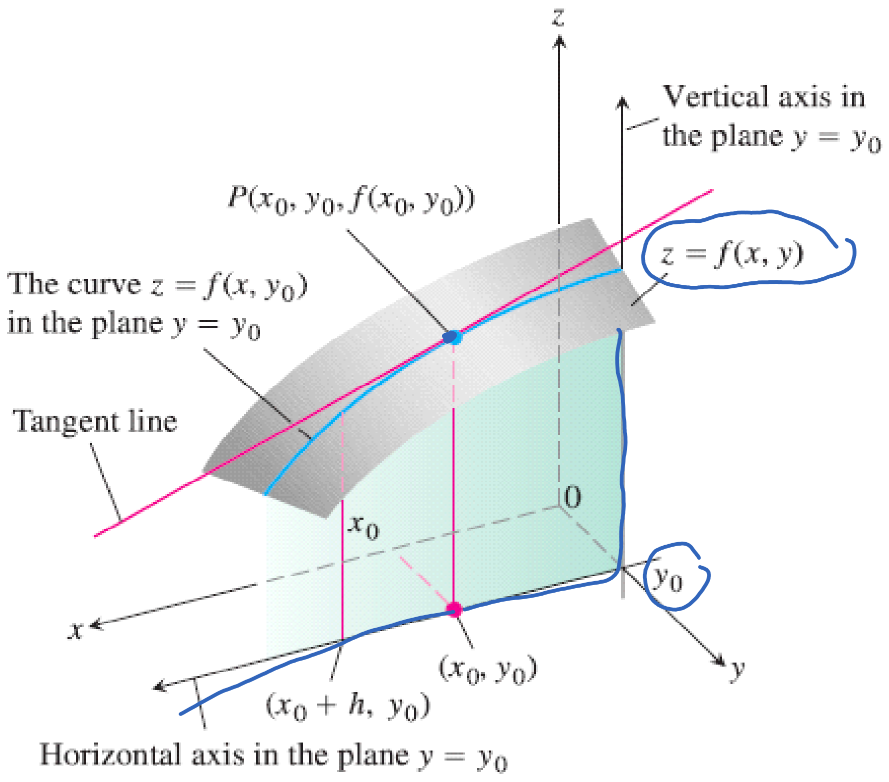
provided the limit exists.

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x:

$$\frac{\partial f}{\partial y}(x_0, y_0), \qquad f_y(x_0, y_0), \qquad \frac{\partial f}{\partial y}, \qquad f_y.$$

$z = f(x,y)$

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0). (In Figure 14.15 this slope is negative.) The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\partial f/\partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0.

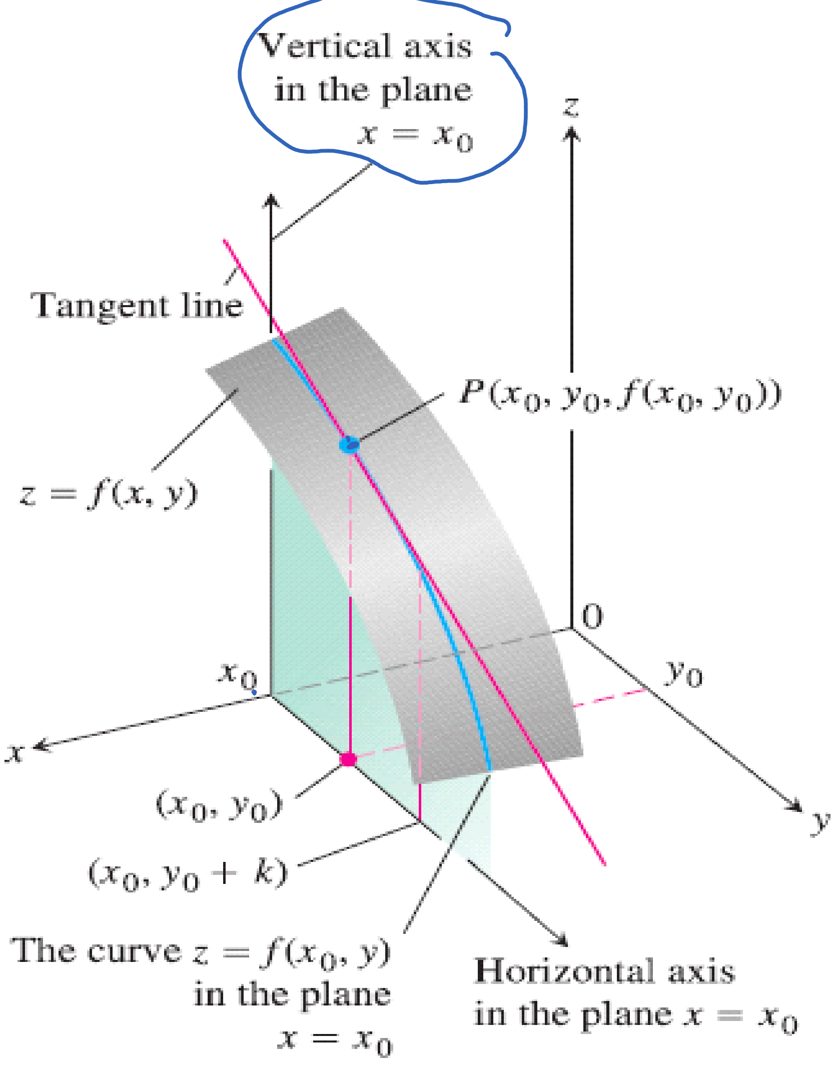


$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$$

FIGURE 14.15 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ (Figure 14.16) is the partial derivative of f with respect to y at (x_0, y_0). The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative gives the rate of change of f with respect to y at (x_0, y_0) when x is held fixed at the value x_0.



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Calculations

Ex. If $f(x,y) = x^2 + xy$. Find, by def'n,

$$\left.\frac{\partial f}{\partial x}\right|_{(4,2)} \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(4,2)}$$

Sol.

$$\left.\frac{\partial f}{\partial x}\right|_{(4,2)} = \lim_{h \to 0} \frac{f(4+h,2) - f(4,2)}{h}$$

$$= \lim_{h \to 0} \frac{(4+h)^2 + (4+h)(2) - 24}{h}$$

$$= \lim_{h \to 0} \frac{16 + 8h + h^2 + 8 + 2h - 24}{h}$$

$$= \lim_{h \to 0} \frac{10h + h^2}{h}$$

$$= \lim_{h \to 0} (10 + 2h) = 10.$$

OR

$$f(x,y) = x^2 + xy$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(xy)$$

$$= 2x + y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,2)} = 2(4) + 2 = 10$$

$$f = x^2 + xy$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,2)} = \lim_{h \to 0} \frac{f(4,2+h) - f(4,2)}{h}$$

$$= \lim_{h \to 0} \frac{16 + 4(2+h) - (16+8)}{h}$$

$$= \lim_{h \to 0} \frac{16 + 8 + 4h - 24}{h}$$

$$= \lim_{h \to 0} 4 = 4 .$$

OR $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(xy)$

$$= 0 + x = x$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,2)} = 4 .$$

Q60) $f(x,y) = \begin{cases} \frac{\sin(x^3+y^4)}{x^2+y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

find $f_x(0,0)$ and $f_y(0,0)$.

Sol.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(h^3)}{h^2}-0}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h^3)}{h^3}$$

$$= \underset{h\to 0}{L} \frac{\cos(h^3)\cdot \cancel{3h^2}}{\cancel{3h^2}}$$

$$= \cos 0 = 1.$$

$f_y(0,0)$ (H.W)

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $\boxed{(4,-5)}$ if

$$f(x,y) = \boxed{x^2} + 3xy + y - 1.$$

Sol. $f_x = 2x + 3y$, $f_y = 0 + 3x + 1$
$= 3x+1$

$f_x(4,-5) = 2(4) + 3(-5) = 8 - 15 = -7$

$f_y(4,-5) = 3(4) + 1 = 13.$

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Sol.

$$\frac{\partial f}{\partial y} = y \frac{\partial}{\partial y}(\sin(xy)) + (\sin(xy)) \frac{\partial}{\partial y}(y)$$

$$= y \cdot \cos(xy) \cdot x + \sin(xy) \cdot 1$$

$$= xy \cos(xy) + \sin(xy).$$

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

$$f_x = \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - (2y)\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - (2y)(-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y \sin x}{(y+\cos x)^2}$$

$$f_y = \frac{(y+\cos x)(2) - (2y)(1)}{(y+\cos x)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}$$

Implicit Differentiation $\qquad z = z(x,y)$

EXAMPLE 4 Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Sol.

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y)$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\cdot\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\therefore \frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}} = \frac{z}{yz-1}$$

ex. $yz - \ln z = x+y$ find $\frac{\partial z}{\partial y}$, where $z = z(x,y)$.

Sol.

$$\frac{\partial}{\partial y}(y z) - \frac{\partial}{\partial y}(\ln z) = \frac{\partial}{\partial y}(x+y)$$

$$y\frac{\partial z}{\partial y} + z\cdot\cancel{\frac{\partial}{\partial y}(y)}^{1} - \frac{1}{z}\frac{\partial z}{\partial y} = 0 + 1$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial y} = 1 - z$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{(1-z)z}{yz-1}$$

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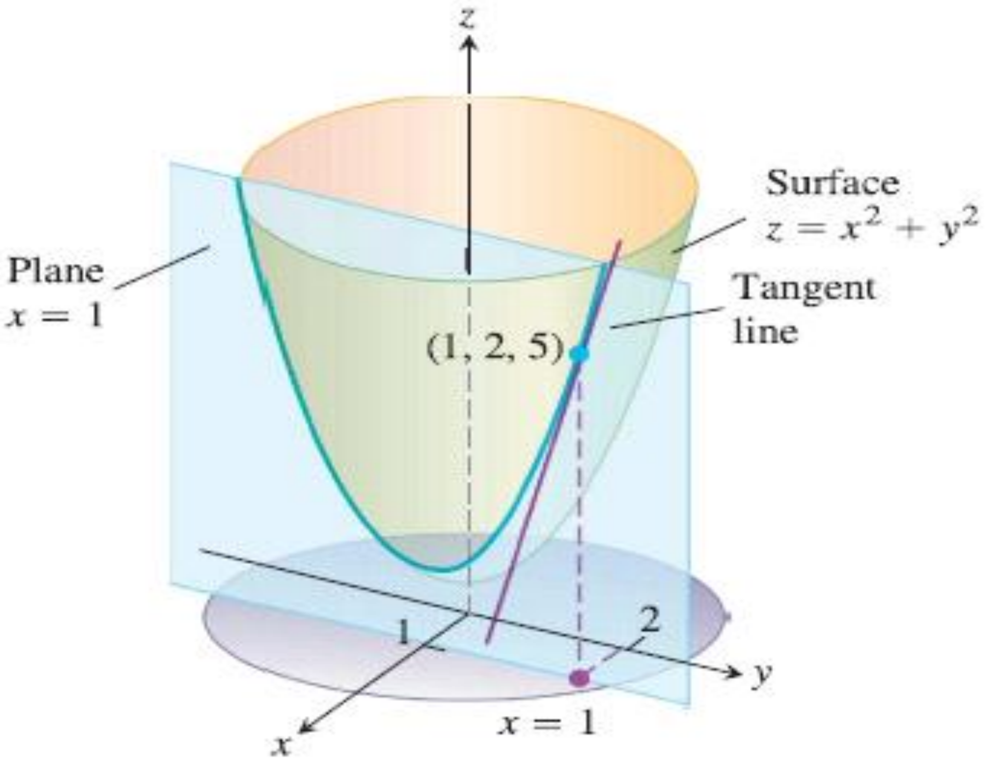
EXAMPLE 5 The plane $\boxed{x = 1}$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5) (Figure 14.18).

Sol. $\text{slope} = \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2)\right|_{(1,2)}$

$= (0 + 2y)\Big|_{(1,2)}$

$= 2(2) = 4$



Functions of More Than Two Variables

EXAMPLE 6 If x, y, and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$ find f_x, f_y, f_z.

$f_x = \frac{\partial}{\partial x}\left(x \sin(y+3z)\right) = \sin(y+3z).$

$f_y = \frac{\partial}{\partial y}\left(x \sin(y+3z)\right) = x \cos(y+3z).(1)$

$$f_z = \frac{\partial}{\partial z}\left(x \sin(y+3z)\right)$$

$$= x\cos(y+3z)\frac{\partial}{\partial z}(y+3z)$$

$$= x\cos(y+3z)\,.\,3$$

$$= 3x\cos(y+3z).$$

EXAMPLE 8 Let

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

$x.x = x^2 \neq 0$

(x,x)

$((x,x))$

$x \to 0$

(Figure 14.20).

(a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.

(b) Prove that f is not continuous at the origin.

(c) Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

ⓐ $$\lim_{\substack{(x,y)\to(0,0)\\ \text{Along } y=x}} f(x,y) = \lim_{(x,y)\to(0,0)} (0) = 0.$$

ⓑ $$f(0,0) = 1 \neq \lim_{(x,y)\to(0,0)} f(x,y)$$

$\therefore$ f is discont. at $(0,0)$.

ⓒ $$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\left(\frac{1-1}{h}\right)$$

$$= \lim_{h\to 0} 0 = 0$$ exists

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{1-1}{k} = 0$$ exists.

Notice that f_x and f_y exist at $(0,0)$ but f is discont. at $(0,0)$.

Second-Order Partial Derivatives

$$z = f(x,y)$$

$$f_{xx}\ ,\ f_{yy}\ ,\ f_{xy}\ ,\ f_{yx}$$

$$w = f(x,y,z)$$

$$f_{xx}\ ,\ f_{yy},\ f_{zz},\ f_{xy},\ f_{yx},\ f_{xz}, f_{zx}$$

$$f_{yz}\ ,\ f_{zy}\ .$$

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}} \text{ or } \boxed{f_{yx},} \qquad \text{and} \qquad \boxed{\frac{\partial^2 f}{\partial y \partial x}} \text{ or } \boxed{f_{xy}.}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$\dfrac{\partial^2 f}{\partial x \partial y}$ Differentiate first with respect to y, then with respect to x.

$\boxed{f_{yx}} = (f_y)_x$ Means the same thing.

$$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{yxx}$$

EXAMPLE 9 If $f(x, y) = x\cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

$$f_{xx} \qquad f_{xy} \qquad f_{yy} \qquad f_{yx}$$

Sol.

$$f_x = \frac{\partial}{\partial x}\left(x\cos y + ye^x\right)$$

$$\boxed{f_x = \cos y + ye^x}$$

$$f_y = \frac{\partial}{\partial y}\left(x\cos y + ye^x\right)$$

$$\boxed{f_y = -x\sin y + e^x}$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\cos y + ye^x\right)$$

$$= ye^x.$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(-x\sin y + e^x\right)$$

$$= -x\cos y.$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\cos y + ye^x\right)$$

$$= \boxed{-\sin y + e^x}.$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(-x\sin y + e^x\right) = \boxed{-\sin y + e^x}$$

The Mixed Derivative Theorem

THEOREM 2—The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

w_{yx} and w_{xy} .

EXAMPLE 10 Find $\partial^2 w/\partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$ ✓

Sol. $$\frac{\partial^2 w}{\partial x \partial y} = w_{yx}$$

$$w_y = x + \frac{(y^2+1)e^y - e^y \cdot 2y}{(y^2+1)^2}$$ ✓

$$w_{yx} = \frac{\partial}{\partial x}(w_y) = \boxed{1}$$

$$\boxed{w_x = y}$$

$$w_{xy} = \frac{\partial}{\partial y}(w_x) = \frac{\partial}{\partial y}(y) = \boxed{1}$$

$\therefore$ $w_{xy} = w_{yx}$, $\forall (x,y)$ since the conditions of Thm 2 hold for w.

Partial Derivatives of Still Higher Order

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} \qquad \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx} .$$

EXAMPLE 11 Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Sol. $f_y = -4xyz + x^2$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(-4xyz + x^2)$$

$$= -4yz + 2x.$$

$$f_{yxy} = \frac{\partial}{\partial y}(f_{yx})$$

$$= \frac{\partial}{\partial y}(-4yz + 2x)$$

$$= -4z$$

$$f_{yxyz} = \frac{\partial}{\partial z}(f_{yxy}) = \frac{\partial}{\partial z}(-4z)$$

$$= -4.$$

Ex. $w = \dfrac{xy}{y^2 + 2\sin^2 y + 1}$ find w_{yxx}.

Sol. $w_{yxx} = w_{xxy}$

$w_x = \dfrac{y}{1 + y^2 + 2\sin^2 y}$, $w_{xx} = 0$

$w_{xxy} = 0.$

Differentiability

THEOREM 4—Differentiability Implies Continuity If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

COROLLARY OF THEOREM 3 If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Ex. Explain why $f(x,y) = 1 + x \cdot \ln(xy-5)$ is diff'ble at $\underbrace{(2,5)}$??

Sol.

$$f_x = x \cdot \frac{y}{xy-5} + \ln(xy-5) \cdot 1$$

$$= \frac{xy}{\boxed{xy-5}} + \ln(xy-5).$$

f_x is cont. at $(2,5)$ since

$$f_x(2,5) = \lim_{(x,y)\to(2,5)} f_x$$

$$f_y = x \frac{x}{xy-5} = \frac{x^2}{xy-5}$$

f_y is cont. at $(2,5)$

Since f_x and f_y are cont. at $(2,5)$

$\Rightarrow f$ is diff'ble at $(2,5)$.

14.4 The Chain Rule

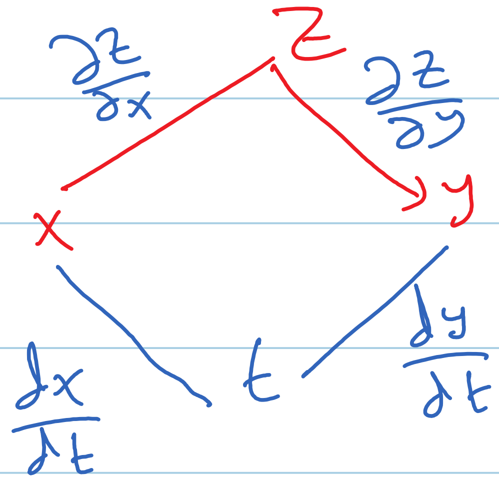
قاعدة السلسلة

Recall, $y = f(t)$, $t = g(x)$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}.$$ Cal 1

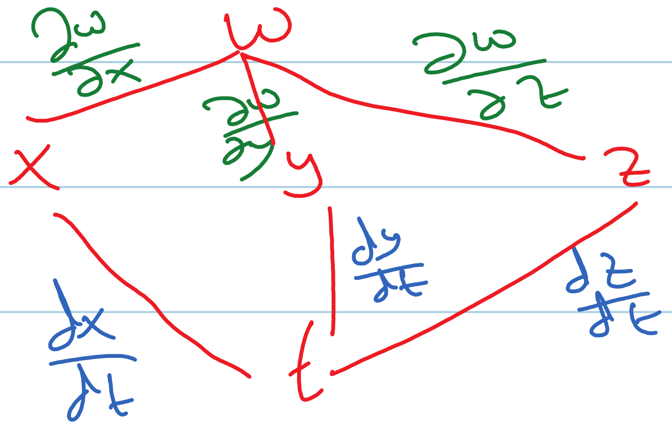
Now, in Cal3, $z = f(x,y)$, $x = x(t)$
$y = y(t)$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$



$w = f(x,y,z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$.

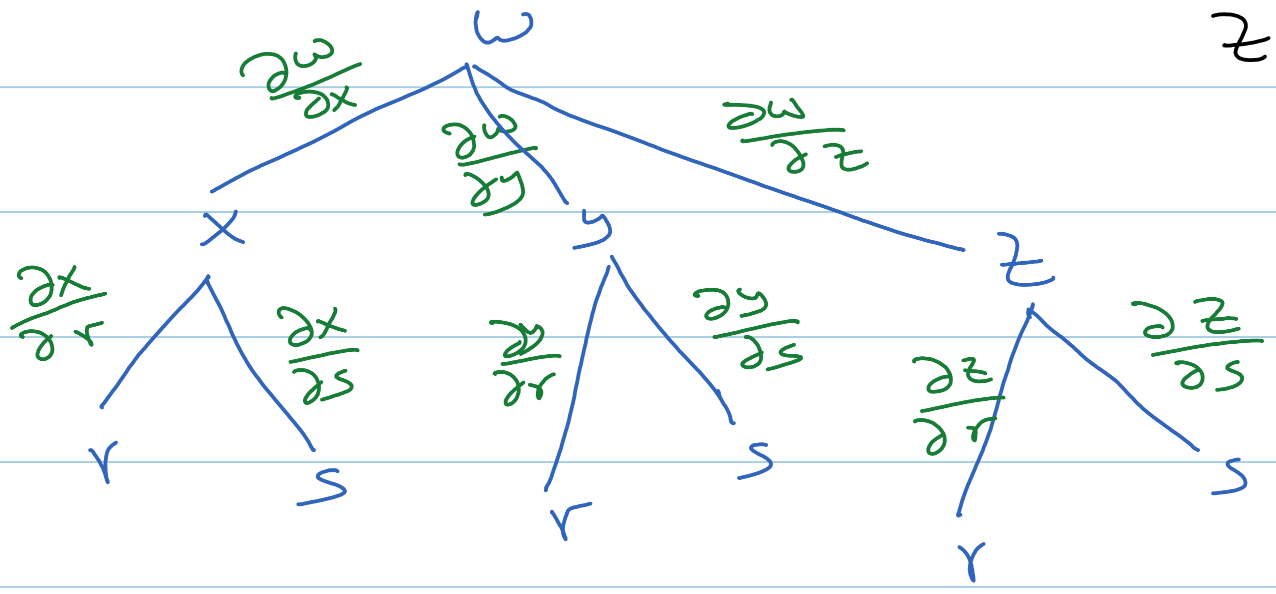
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$



$w = w(x,y,z)$, $x = x(r,s)$

$y = y(r,s)$

$z = z(r,s)$

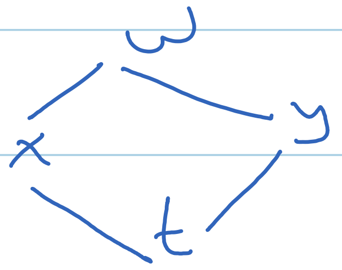


$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Ex1. Let $\boxed{w = xy}$, $x = \cos t$, $y = \sin t$

Find $\frac{dw}{dt}\Big|_{\boxed{t=\pi/2}}$



Sol. $\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$

$= y\cdot(-\sin t) + x\cdot(\cos t)$

$\therefore \frac{dw}{dt}\Big|_{t=\frac{\pi}{2}, x=0, y=1} = 1\left(-\sin\frac{\pi}{2}\right) + 0\cdot\cos\frac{\pi}{2}$

$= -1$.

$t = \frac{\pi}{2}$

$x = \cos\frac{\pi}{2} = 0$

$y = \sin\frac{\pi}{2} = 1$

Ex② $w = xy + z$, $x = \cos t$

$y = \sin t$, $z = t$

Find $\frac{dw}{dt}$ at $t = 0$

Sol. $t = 0 \Rightarrow x = 1$, $y = 0$, $z = 0$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= y(-\sin t) + x(\cos t) + 1 \cdot 1$$

$$= (\sin t)(-\sin t) + \cos t \cdot \cos t + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

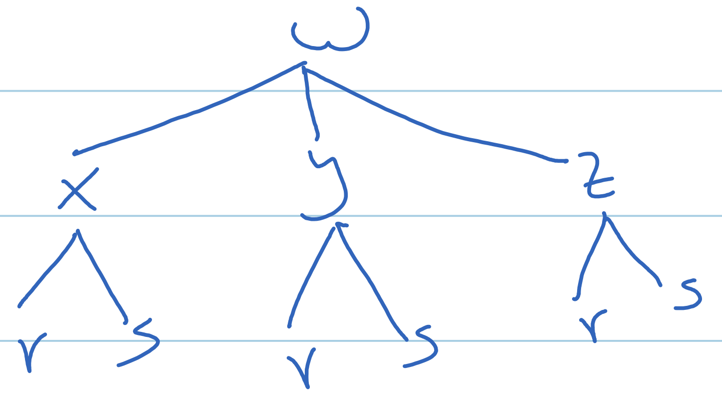
$$= \cos(2t) + 1 .$$

$$\therefore \left.\frac{dw}{dt}\right|_{t=0} = \cos 0 + 1 = 2 .$$

EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$



$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \cancelto{0}{\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}}$$

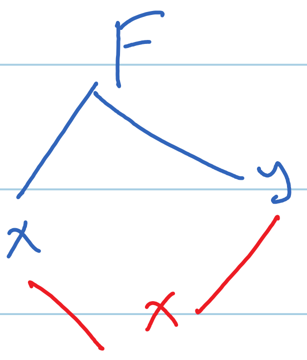
$$= (1)\left(\frac{-r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)$$

$$= \frac{-r}{s^2} + \frac{2}{s} = \frac{2s - r}{s^2}$$

Implicit Differentiation Revisited

If $F(x,y) = 0$ find $\frac{dy}{dx}$.

$$\frac{\partial F}{\partial x}\cdot\boxed{\frac{dx}{dx}}^{1} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$



$$\Longrightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}, \quad F_y \neq 0$$

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THEOREM 8—A Formula for Implicit Differentiation Suppose that $F(x, y)$ is differentiable and that the equation $\boxed{F(x, y) = 0}$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \tag{1}$$

EXAMPLE 5 Use Theorem 8 to find dy/dx if $y^2 - x^2 - \sin xy = 0$.

Sol. Cal 1

$$2y\frac{dy}{dx} - 2x - \cos(xy)\left[x\frac{dy}{dx} + y\cdot 1\right] = 0$$

$$\underbrace{2y\frac{dy}{dx}} - 2x - \underbrace{x\cos(xy)\frac{dy}{dx}} - y\cos(xy) = 0$$

$$\left[2y - x\cos(xy)\right]\frac{dy}{dx} = 2x + y\cos(xy)$$

$$\frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)} \quad ✓$$

Cal 3 (Thm 8) $\quad y^2 - x^2 - \sin(xy) = 0$

$$F(x,y) = y^2 - x^2 - \sin(xy)$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - \cos(xy)\cdot y}{2y - \cos(xy)\cdot x}$$

$$= \frac{2x + y\cos(xy)}{2y - x\cos(xy)}.$$

$$F(x,y,z)=0, \qquad z=z(x,y)$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

$$F(x,y,z)$$

EXAMPLE 6 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at (0, 0, 0) if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0,0)}=-\left.\frac{F_x}{F_z}\right|_{(0,0,0)}=-\frac{3x^2+yze^{xz}}{2z+yxe^{xz}+\cos y}$$

$$=-\frac{0+0}{0+0+1}=0$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,0,0)}=-\left.\frac{F_y}{F_z}\right|_{(0,0,0)}=-\frac{e^{xz}+-z\sin y}{2z+yxe^{xz}+\cos y}$$

$$=-\frac{1+0}{0+0+1}=-1$$

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H.w If $z^3 - xz - y = 0$, where z defines a function of x and y.

Find $\dfrac{\partial^2 z}{\partial x \partial y}$

14.5 Directional Derivatives and Gradient Vectors

Directional Derivatives in the Plane

$\left(D_{\vec{u}} f\right)_{P_0}$

DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(D_{\vec{u}} f\right)_{P_0} \qquad \left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \tag{1}$$

provided the limit exists.

The **directional derivative** defined by Equation (1) is also denoted by

$$(D_{\mathbf{u}} f)_{P_0}.$$

"The derivative of f at P_0 in the direction of $\mathbf{u}$"

EXAMPLE 1 Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(\overset{x_0,\,y_0}{1, 2})$ in the direction of the unit vector $\mathbf{u} = \underset{u_1}{(1/\sqrt{2})}\mathbf{i} + \underset{u_2}{(1/\sqrt{2})}\mathbf{j}$.

$$D_{\vec{u}} f\Big|_{P_0(1,2)} = \lim_{h\to0} \frac{f\left(1+\frac{1}{\sqrt2}h,\ 2+\frac{1}{\sqrt2}h\right) - f(1,2)}{h}$$

$$= \lim_{h\to0} \frac{\left(1+\frac{1}{\sqrt2}h\right)^2 + \left(1+\frac{1}{\sqrt2}h\right)\left(2+\frac{1}{\sqrt2}h\right) - (1+2)}{h}$$

$$= \lim_{h\to0} \frac{1+\sqrt2 h + \frac12 h^2 + 2 + \frac{3}{\sqrt2}h + \frac12 h^2 - 3}{h}$$

$$= \lim_{h\to0} \frac{h^2 + \left(\sqrt2 + \frac{3}{\sqrt2}\right)h}{h}$$

$$= \lim_{h\to0} \left[h + \left(\sqrt2 + \frac{3\sqrt2}{2}\right)\right] = \frac{5\sqrt2}{2}$$

the rate of change of f

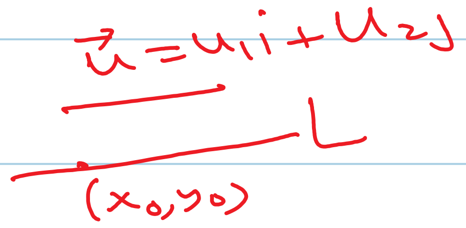
$f(x,y) = x^2+xy$ at $P_0(1,2)$

in the direction $\vec{u} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

is $\frac{5}{\sqrt{2}}$

That is, $\left(D_{\vec{u}} f\right)_{P_0} = \frac{5}{\sqrt{2}}$.

Calculation and Gradients



Line $\quad x = x_0 + u_1 t, \quad y = y_0 + u_2 t$

$$\left(\frac{df}{ds}\right)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{dy}{ds}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0} i + \left(\frac{\partial f}{\partial y}\right)_{P_0} j\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{(u_1 i + u_2 j)}_{\text{Direction } \vec{u}}$$

$(\nabla f)_{P_0}$

∇f: nabla f

∇f : del(f)
grad(f)

$$\therefore \left(D_u f\right)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f.

THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$(D_{\vec{u}}f)_{P_0} = \left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \qquad (4)$$

the dot product of the gradient ∇f at P_0 and $\mathbf{u}$.

EXAMPLE 2 Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Sol. $(D_{\vec{v}}f)_{P_0}$.

the direction $\vec{v}$ is not unit ??

$$\boxed{\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{3}{5}i - \frac{4}{5}j} \quad , \quad |\vec{v}| = \sqrt{9+16} = 5$$

$$f_x\Big|_{(2,0)} = \left(e^y - \sin(xy)\cdot y\right)\Big|_{(2,0)} = 1 - 0 = \boxed{1}$$

$$f_y\Big|_{(2,0)} = \left(xe^y - x\sin(xy)\right)\Big|_{(2,0)} = 2 - 0 = \boxed{2}$$

$$\boxed{\therefore (\nabla f)_{P_0=(2,0)} = 1i + 2j = i + 2j}$$

$$\therefore (D_{\vec{u}}f)_{P_0} = (\nabla f)_{P_0}\cdot\vec{u} = \left(\frac{3}{5}\right)(1) - \frac{4}{5}(2) = -1$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = |\nabla f|\cancel{|\vec{u}|}^{1}\cos\theta$$
$$= |\nabla f| \cos\theta$$

Evaluating the dot product in the formula

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f, reveals the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f|\cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f|\cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f|\cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1).

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Sol.

$$f_x\big|_{(1,1)} = x\big|_{(1,1)} = 1 \quad , \quad f_y\big|_{(1,1)} = y\big|_{(1,1)} = 1$$

$$\therefore \nabla f\big|_{(1,1)} = i + j \quad , \quad |\nabla f| = \sqrt{2}.$$

ⓐ $$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$$

$$D_{\vec{u}}f\big|_{(1,1)} = |\nabla f| = \sqrt{2}.$$

ⓑ $$\vec{u} = \frac{-\nabla f}{|\nabla f|} = -\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

$$D_{\vec{u}}f\big|_{(1,1)} = -|\nabla f| = -\sqrt{2}$$

ⓒ $D_{\vec{u}} f \Big|_{(1,1)} = 0$

$$(\nabla f)_{(1,1)} \cdot \vec{u} = 0$$

$$(i+j) \cdot \vec{u} = 0$$

$$(i+j) \cdot (u_1 i + u_2 j) = 0$$

$u_1 + u_2 = 0$ and $u_1^2 + u_2^2 = 1$

$\boxed{u_1 = -u_2}$ $\therefore 2u_1^2 = 1$

$$u_1 = \pm \frac{1}{\sqrt{2}}$$

$u_1 = \frac{1}{\sqrt{2}} \Rightarrow u_2 = -\frac{1}{\sqrt{2}} \Rightarrow \vec{u} = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$

$u_1 = -\frac{1}{\sqrt{2}} \Rightarrow u_2 = \frac{1}{\sqrt{2}} \Rightarrow \vec{V} = -\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

تحقق

$$\left(D_{\vec{u}} f\right)_{(1,1)} = (\nabla f)\Big|_{(1,1)} \cdot \vec{u}$$

$$= (i+j) \cdot \left(\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

and $D_{\vec{V}} f = \nabla f \Big|_{(1,1)} \cdot \vec{V}$

$$= (i+j) \cdot \left(-\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j\right) = 0$$

Gradients and Tangents to Level Curves

If $f(x,y)$ has a constant c along a smooth curve

$$\vec{r} = g(t)i + h(t)j$$

$$\Rightarrow \quad f(x,y) = c$$ level Curve

$$f(g(t), h(t)) = c$$

$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t) = 0$$

$$\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j\right) \cdot \left(g'(t)i + h'(t)j\right) = 0$$

$$\nabla f \cdot \frac{d\vec{r}}{dt} = 0$$

∇f is normal to the tangent vector $\frac{d\vec{r}}{dt}$

$\Rightarrow$ ∇f is normal to the curve

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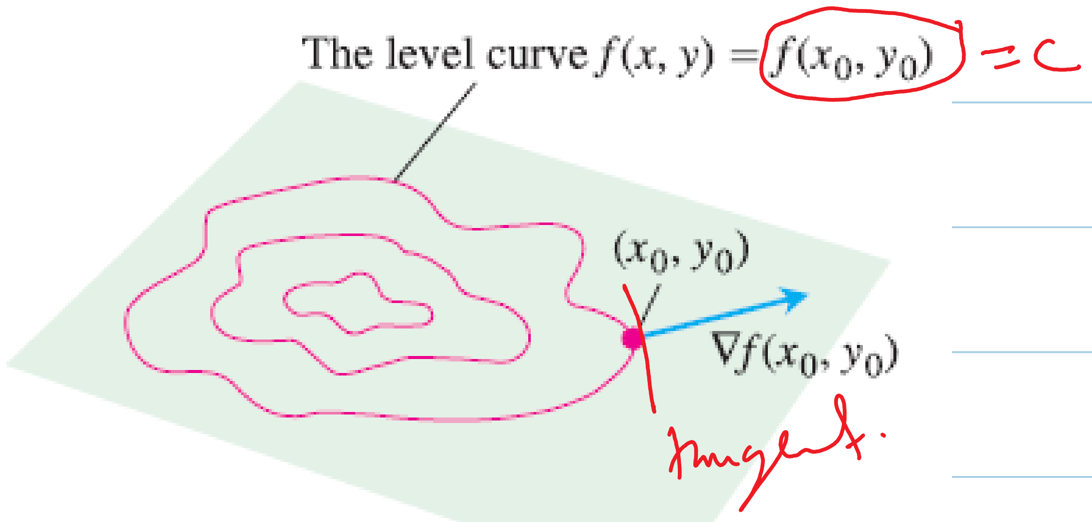
At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.30).



The line through $P_0(x_0,y_0)$ normal to the vector $\vec{N} = Ai + Bj$ has the eq. $A(x-x_0)+B(y-y_0)=0$

If $\vec{N} = (\nabla f)_{(x_0,y_0)} = f_x(x_0,y_0)\, i + f_y(x_0,y_0)\, j$ the eq. is the tangent line given by

$$\boxed{f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0) = 0}$$

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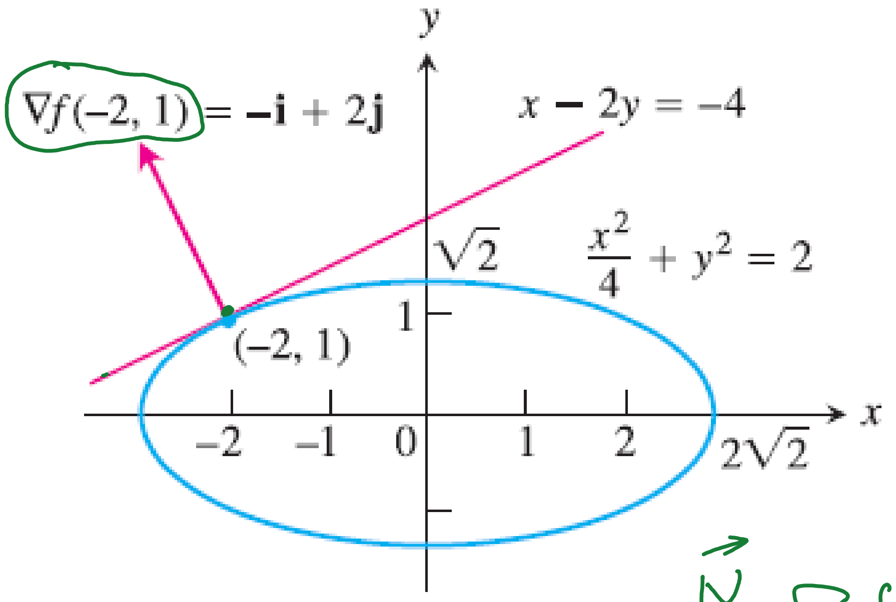
EXAMPLE 4 Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2$$

(Figure 14.31) at the point $(-2, 1)$.

$f(x,y) = 2$ level curve



$f(x,y) = \frac{x^2}{4} + y^2$

$f_x \Big|_{(-2,1)} = \frac{2x}{4} \Big|_{(-2,1)} = -1$

$f_y \Big|_{(-2,1)} = 2y \Big|_{(-2,1)} = 2$

$\therefore \vec{N} = \nabla f \Big|_{(-2,1)} = -i + 2j$

$\therefore$ The tangent line

$f_x(-2,1)(x+2) + f_y(-2,1)(y-1) = 0$

$-1(x+2) + 2(y-1) = 0$

$$\boxed{-x + 2y = 4}$$

Functions of Three Variables

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||u|\cos\theta = |\nabla f|\cos\theta,$$

EXAMPLE 6

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Sol. ⓐ $|\vec{v}| = \sqrt{4+9+36} = 7$

$$\boxed{\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k}$$

$$f_x\Big|_{(1,1,0)} = 3x^2 - y^2\Big|_{(1,1,0)} = 3-1 = 2$$

$$f_y(1,1,0) = -2xy\Big|_{(1,1,0)} = -2$$

$$f_z(1,1,0) = -1\Big|_{(1,1,0)} = -1$$

$$\therefore \boxed{\nabla f\Big|_{P_0} = 2i - 2j - k}$$

$$\therefore D_{\vec{v}}f\Big|_{P_0} = (\nabla f)_{P_0} \cdot \vec{u} = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$$

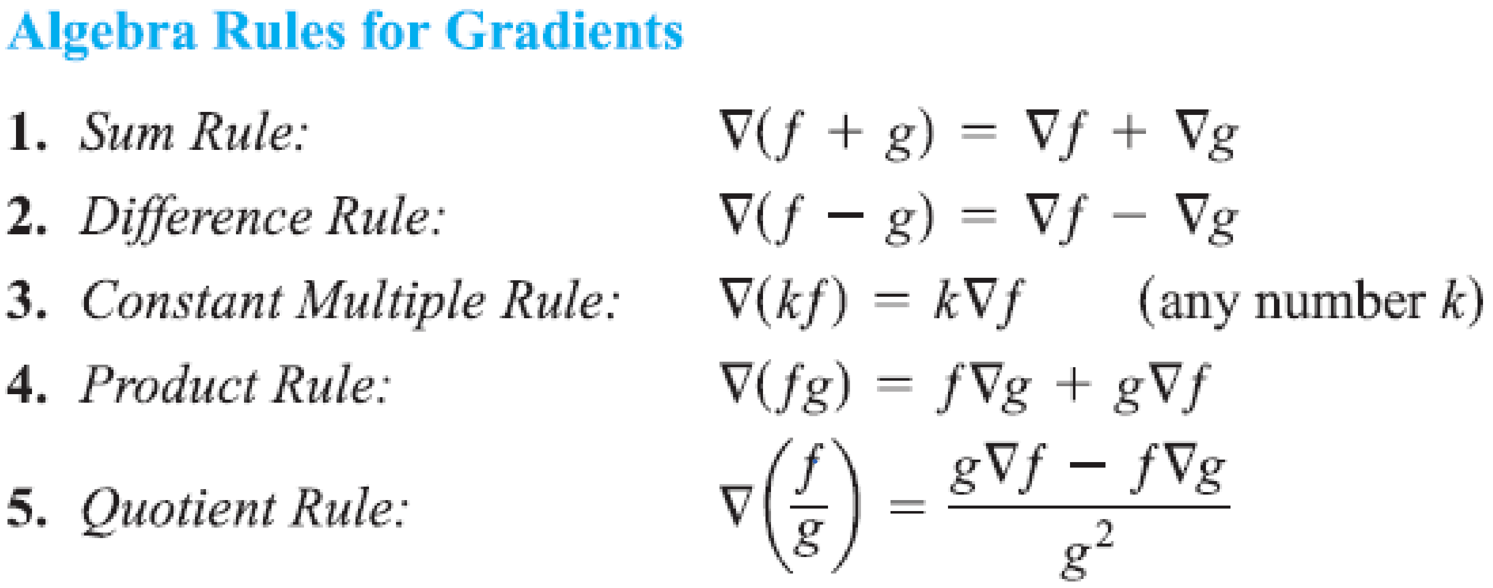
ⓑ f increases most rapidly in the direction of $\nabla f = 2i - 2j - k$

The rate of change is $|\nabla f| = \sqrt{9} = 3$

f decreases most rapidly in the direction of $-\nabla f = -2i + 2j + k$

the rate of change is

$$-|\nabla f| = -\sqrt{9} = -3$$



Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

12.5 $\vec{N}$, P

14.6 Tangent Planes and Differentials

Tangent Planes and Normal Lines

$f(x,y,z)=c$

If $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = c$. Differentiating both sides of this equation with respect to t leads to

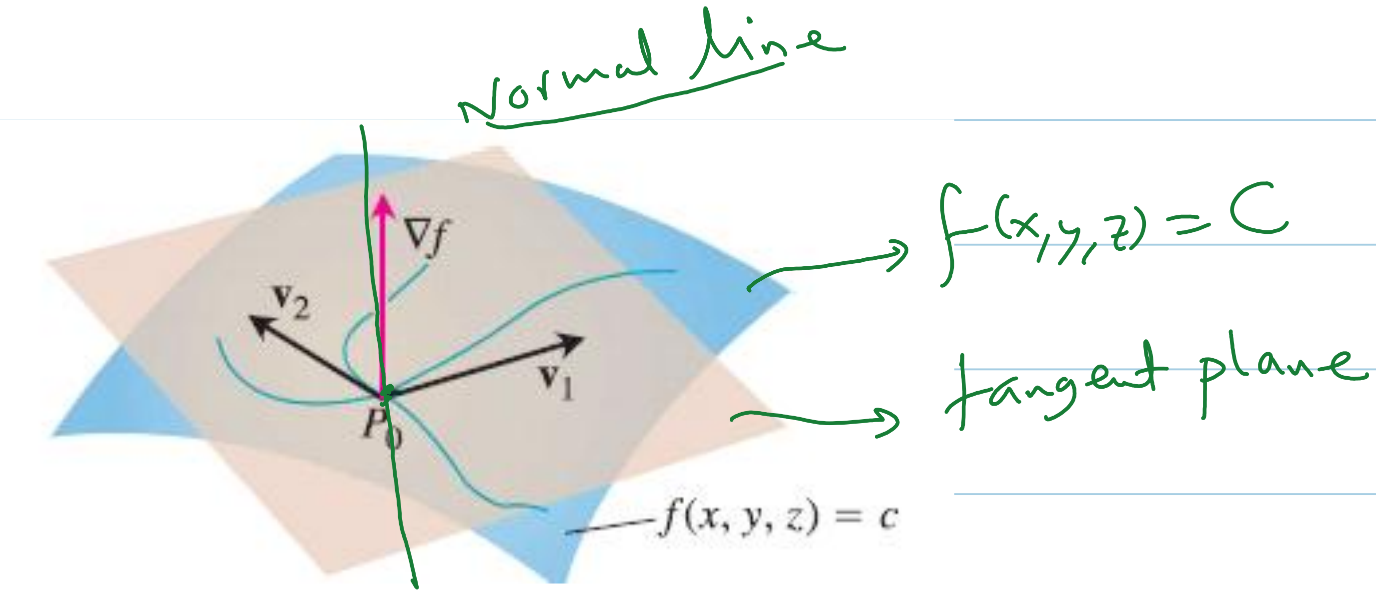
$$\frac{d}{dt} f(g(t), h(t), k(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0 \qquad \text{Chain Rule}$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j} + \frac{dk}{dt}\mathbf{k}\right)}_{d\mathbf{r}/dt} = 0.$$

DEFINITIONS The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.



Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t$$

EXAMPLE 1 Find the tangent plane and normal line of the surface

$$f(x, y, z) = \boxed{x^2 + y^2 + z - 9 = 0} \longrightarrow z = 9 - x^2 - y^2$$

A circular paraboloid

at the point $P_0(1, 2, 4)$.

Sol. $f_x\big|_{P_0} = 2x\big|_{(1,2,4)} = 2$

$f_y\big|_{P_0} = 2y\big|_{(1,2,4)} = 4$

$f_z\big|_{P_0} = 1\big|_{P_0} = 1$

$$\boxed{\nabla f\big|_{P_0} \quad 2i + 4j + k}$$

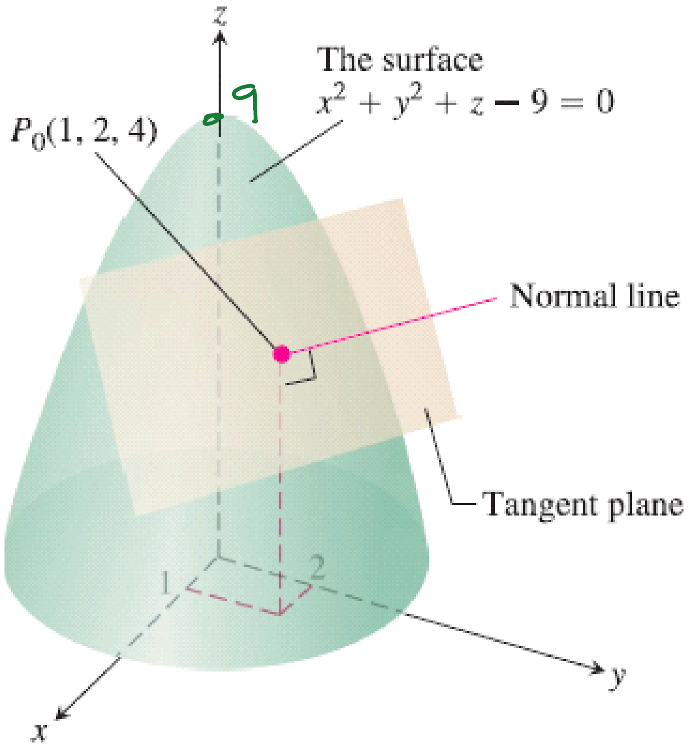


FIGURE 14.33 The tangent plane and normal line to this surface at P_0 (Example 1).

the tangent plane is

$f_x(P_0)(x-1) + f_y(P_0)(y-2) + f_z(P_0)(z-z_0) = 0$

$\Rightarrow \quad 2(x-1) + 4(y-2) + 1(z-4) = 0$

or $\boxed{2x + 4y + z = 14}$

The line normal to the surface at P_0 is

$x = x_0 + f_x(P_0)t = 1 + 2t$

$y = y_0 + f_y(P_0)t = 2 + 4t \quad , t \in \mathbb{R}.$

$z = z_0 + f_z(P_0)t = 4 + t$

$F(x,y,z) = f(x,y) - z = 0$ $\quad f(x,y,z) = c$

Plane Tangent to a Surface $\boxed{z = f(x, y)}$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \boxed{-} (z - z_0) = 0. \qquad (4)$$

EXAMPLE 2 Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.

Sol. $f(x,y,z) = x\cos y - y e^x - z = 0$

$$\nabla f\big|_{(0,0,0)} = \left(f_x i + f_y j + f_z k\right)_{P_0(0,0,0)}$$

$$= (\cos y - ye^x) i + (-x\sin y - e^x) j - k \Big|_{(0,0,0)}$$

$$= i - j - k$$

The tangent plane is

$$1(x-0) - 1(y-0) - 1(z-0) = 0$$

or $\quad x - y - z = 0.$

EXAMPLE 3 The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (Figure 14.34). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

The tangent line is $\perp$ ∇g and ∇f at P_0

$\Rightarrow$ Tangent line $\parallel$ $\vec{v} = \nabla f \times \nabla g$ at P_0.

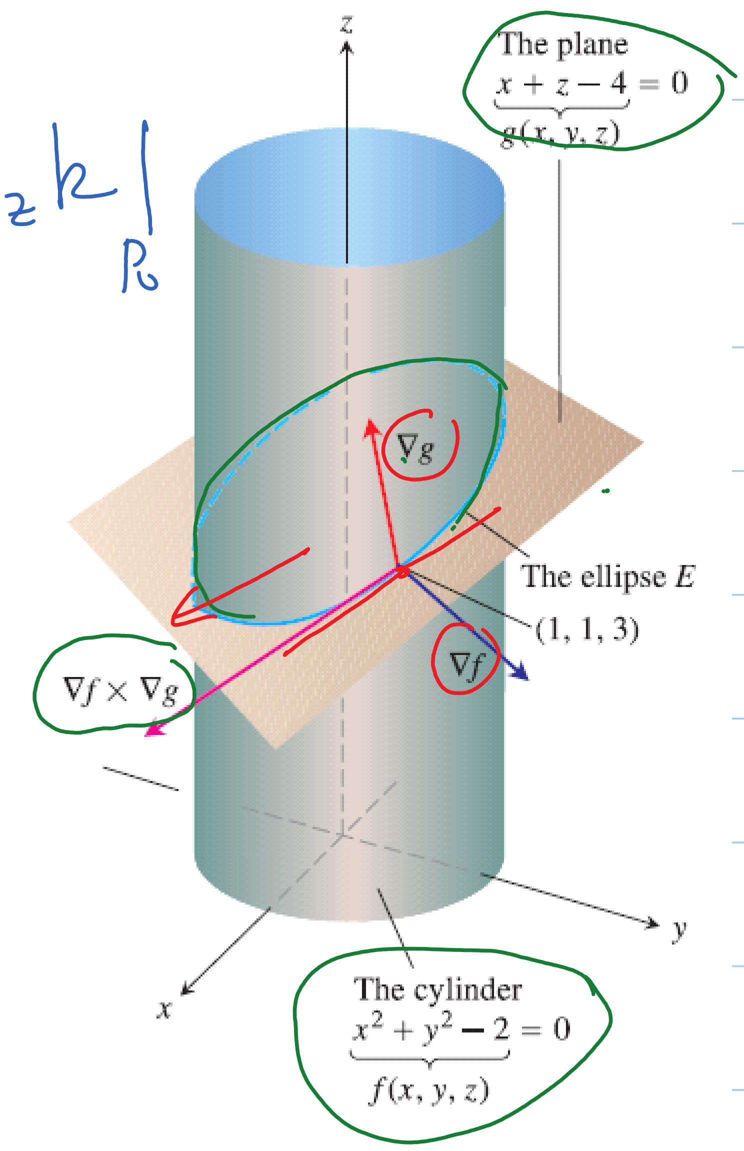
$$\nabla f\big|_{P_0} = f_x i + f_y j + f_z k \big|_{P_0}$$

$$= 2x i + 2y j \big|_{(1,1,3)} = 2i + 2j$$

$$\nabla g\Big|_{P_0} = g_x i + g_y j + g_z k\Big|_{P_0}$$
$$= i + k\Big|_{P_0}$$
$$= i + k$$

$\vec{v} = \nabla f \times \nabla g$ at P_0

$$= \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$



$\vec{v} = 2i - 2j - 2k$, $P_0(1, 1, 3)$

the tangent line is

$x = 1 + 2t$, $y = 1 - 2t$, $z = 3 - 2t$.

Estimating Change in a Specific Direction

Recall, $y = f(x) \Rightarrow df = f'(P_0)\,ds$ (ordinary derv.) (increment)

ds: small distance from a point P_0 to another point.

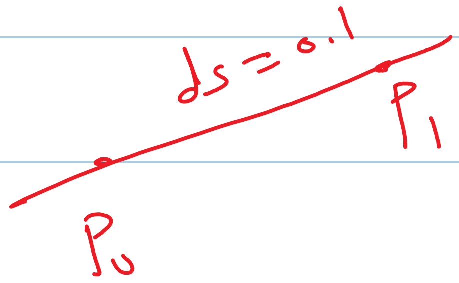
If f is a function of two or more variables,

$df = \left(\nabla f\big|_{P_0} \cdot \vec{u}\right) ds$. (Directional derivative) X increment

Estimating the Change in *f* in a Direction u

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{\left(\nabla f|_{P_0} \cdot \mathbf{u}\right)}_{\text{Directional derivative}} \underbrace{ds}_{\text{Distance increment}}$$



EXAMPLE 4 Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Sol. $\boxed{ds = 0.1}$, $\vec{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{2i+j-2k}{\sqrt{4+1+4}}$

$$= \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k$$

$$\nabla f\Big|_{P_0} = f_x i + f_y j + f_z k \Big|_{P_0}$$

$$= (y\cos x)\, i + (\sin x + 2z)\, j + (2y)\, k \Big|_{(0,1,0)}$$

$$= i + 2k$$

$$\therefore df = \left(\nabla f\Big|_{P_0} \cdot \vec{u}\right) ds$$

$$= \left[(i+2k) \cdot \left(\frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k\right)\right](0.1)$$

$$= \left(\frac{2}{3} - \frac{4}{3}\right)(0.1) = \frac{-0.2}{3}$$

$$= -\frac{2}{30}$$

$$\approx -0.067 \text{ unit.}$$

How to Linearize a Function of Two Variables

Recall, $y = f(x) \Rightarrow$ linearization at $x = x_0$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$\therefore L(x) \approx f(x)$$

DEFINITIONS The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{5}$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

EXAMPLE 5 Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point (3, 2).

$$\boxed{L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)}$$

$$f(3,2) = 3^2 - 6 + \frac{1}{2}(4) + 3 = 8$$

$$f_x(3,2) = (2x - y)\Big|_{(3,2)} = 2(3) - 2 = 4$$

$$f_y(3,2) = (-x + y)\Big|_{(3,2)} = -3 + 2 = -1$$

$$\therefore L(x,y) = 8 + 4(x-3) - 1(y-2)$$

$$\boxed{L(x,y) = 4x - y - 2}$$

Ex. Approximate $f(2.9, 1.9)$ in the last ex.

Sol. $f(2.9, 1.9) \approx L(2.9, 1.9) = 4(2.9) - 1.9 - 2$

$= 11.6 - 3.9 = 7.7$

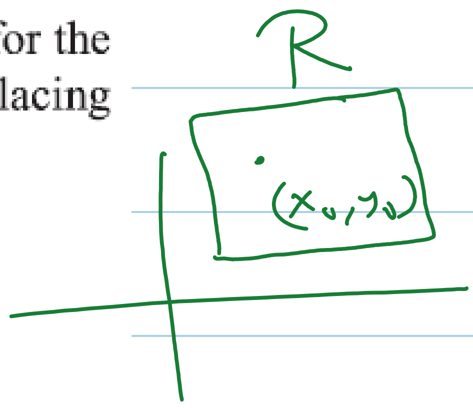
The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \le \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$



EXAMPLE 6 Find an upper bound for the error in the approximation $f(x, y) \approx L(x, y)$ in Example 5 over the rectangle

$$R: \quad |x - \overset{x_0}{3}| \le 0.1, \qquad |y - \overset{y_0}{2}| \le 0.1. \ ✓$$

Express the upper bound as a percentage of $f(\underset{x_0}{3}, \underset{y_0}{2})$, the value of f at the center of the rectangle.

Sol. $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$

$f_x = 2x - y, \quad f_y = -x + y$

$f_{xx} = 2, \quad f_{yy} = 1, \quad f_{xy} = -1$

$|f_{xx}| = 2, \quad |f_{yy}| = 1, \quad |f_{xy}| = 1$

the largest of $\{|f_{xx}|, |f_{yy}|, |f_{xy}|\}$ is 2

$\therefore M = 2$

$$|E(x,y)| \le \frac{1}{2} M (|x - x_0| + |y - y_0|)^2$$

$$= \frac{1}{2} M (|x - 3| + |y - 2|)^2$$

$$\le \frac{1}{2}(2)(0.1 + 0.1)^2 = 0.04$$

As a percentage of $f(3,2) = 8$, the error is no greater than $= \frac{0.04}{8} \times 100\% = 0.5\%$

Functions of More Than Two Variables

1. The **linearization** of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).$$

2. Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R. Then the **error** $E(x, y, z) = f(x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality

$$|E| \le \frac{1}{2} M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

EXAMPLE 10 Find the linearization $L(x, y, z)$ of

$$f(x, y, z) = x^2 - xy + 3 \sin z$$

at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangle

$$R: \quad |x - \underset{x_0}{\boxed{2}}| \le 0.01, \qquad |y - \underset{y_0}{\boxed{1}}| \le 0.02, \qquad \underset{z_0 = 0}{|z|} \le 0.01.$$

Sol.

$$f(2,1,0) = 4 - 2(1) + 3\sin 0 = \boxed{2}$$

$$f_x(2,1,0) = (2x - y)\Big|_{(2,1,0)} = 4 - 1 = \boxed{3}$$

$$f_y(2,1,0) = -x\Big|_{(2,1,0)} = \boxed{-2}$$

$$f_z(2,1,0) = 3\cos z\Big|_{(2,1,0)} = 3\cos 0 = \boxed{3}$$

$$L(x,y,z) = f(2,1,0) + f_x(2,1,0)(x-2) + f_y(2,1,0)(y-1) + f_z(2,1,0)\,z$$

$$= 2 + 3(x-2) - 2(y-1) + 3z$$

$$= 3x - 2y + 3z - 2$$

$$|E(x,y,z)| \le \frac{1}{2} M \left(|x-x_0| + |y-y_0| + |z-z_0| \right)^2$$

$$= \frac{1}{2} M \left(|x-2| + |y-1| + |z| \right)^2$$

Now, $f_x = 2x-y$, $f_y = -x$, $f_z = 3\cos z$

$f_{xx} = 2$, $f_{yy} = 0$, $f_{zz} = -3\sin z$

$f_{xy} = -1$, $f_{xz} = 0$, $f_{yz} = 0$

$-0.01 < z < 0.01$

$|f_{xx}| = \boxed{2}$, $|f_{yy}| = \boxed{0}$, $|f_{zz}| = 3|\sin z|$

$< 3\sin(0.01)$

$\approx \boxed{0.03}$

$|f_{xy}| = \boxed{1}$, $|f_{xz}| = \boxed{0}$, $|f_{yz}| = \boxed{0}$

$\therefore$ $M = 2$ (upper bound of the second partials deriv.)

$$\therefore \quad |E| \le \frac{1}{2}(2)\left(|x-2| + |y-1| + |z| \right)^2$$

$$\le \left(0.01 + 0.02 + 0.01 \right)^2 = 0.0016.$$

As a percentage of $f(2,1,0) = 2$, the error no greater than

$$\frac{0.0016}{2} \times 100\% = 0.08\%$$

Differentials

Cal I

Recall, $y = f(x)$

x changes from a to $a+\Delta x$

$$\Rightarrow \Delta f = f(a+\Delta x) - f(a)$$

the differental of f is given

$$\boxed{df = f'(a)\,\Delta x}$$

DEFINITION If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

in the linearization of f is called the **total differential of f**.

If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, and z_0 by small amounts dx, dy, and dz, the **total differential**

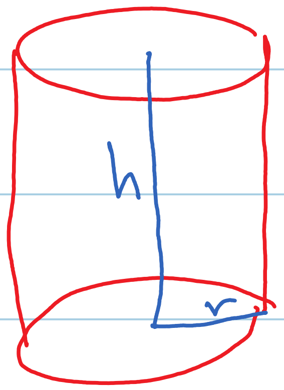
$$df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$

gives a good approximation of the resulting change in f.

EXAMPLE 7 Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Volume $= \pi r^2 h$

$$\boxed{V(r,h) = \pi r^2 h}$$



Given $r=1$, $h=5$, $dr=+0.03$, $dh=-0.1$

$$dV = V_r\,dr + V_h\,dh$$

$$= (2\pi r h)\,dr + (\pi r^2)\,dh$$

$$= 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1)$$

$$= 0.3\pi - 0.1\pi = 0.2\pi.$$

$$V = \pi r^2 h = \pi(1)^2(5) = 5\pi$$

Percentage error in the Calculation of V is $\left|\frac{dV}{V}\right| \times 100\% = \left|\frac{0.2\pi}{5\pi}\right| \times 100\%$

$$= 4\%$$

EXAMPLE 9 The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V.

Given.

$$V = \pi r^2 h$$

$$\left|\frac{dr}{r}\right| \leq 2\% \quad , \quad \left|\frac{dh}{h}\right| \leq 0.5\%$$

find $\left|\frac{dV}{V}\right| \leq ??$

Now, $$\left|\frac{dV}{V}\right| = \left|\frac{V_r\,dr + V_h\,dh}{\pi r^2 h}\right|$$

$$= \left|\frac{2\pi r h\,dr + \pi r^2\,dh}{\pi r^2 h}\right|$$

$$\boxed{=} \left|\frac{2\cancel{\pi} r \cancel{h}\,dr}{\cancel{\pi} r^2 \cancel{h}} + \frac{\cancel{\pi} \cancel{r^2}\,dh}{\cancel{\pi} \cancel{r^2} h}\right|$$

$$|a+b| \leq |a| + |b|$$

تذكر .

$$\leq 2\left|\frac{dr}{r}\right| + \left|\frac{dh}{h}\right|$$

$$\leq 2(2\%) + 0.5\%$$

$$= 4.5\% = 0.045$$

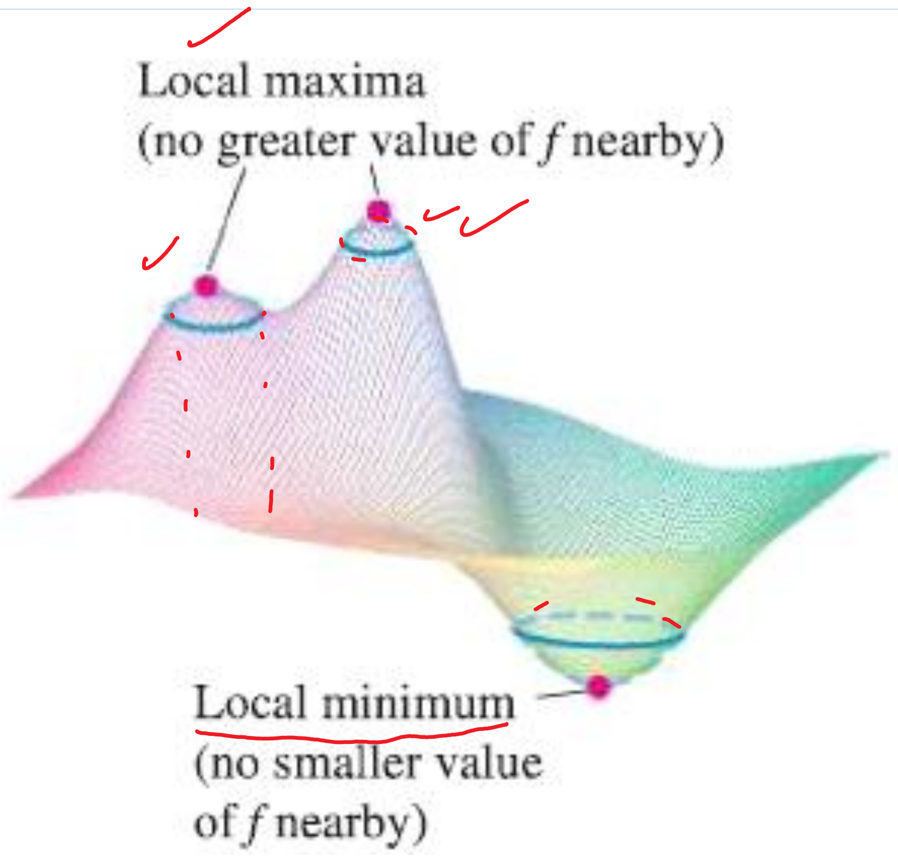
the error in Volume Calculation is at amost 4.5% .

14.7 Extreme Values and Saddle Points

Derivative Tests for Local Extreme Values

DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

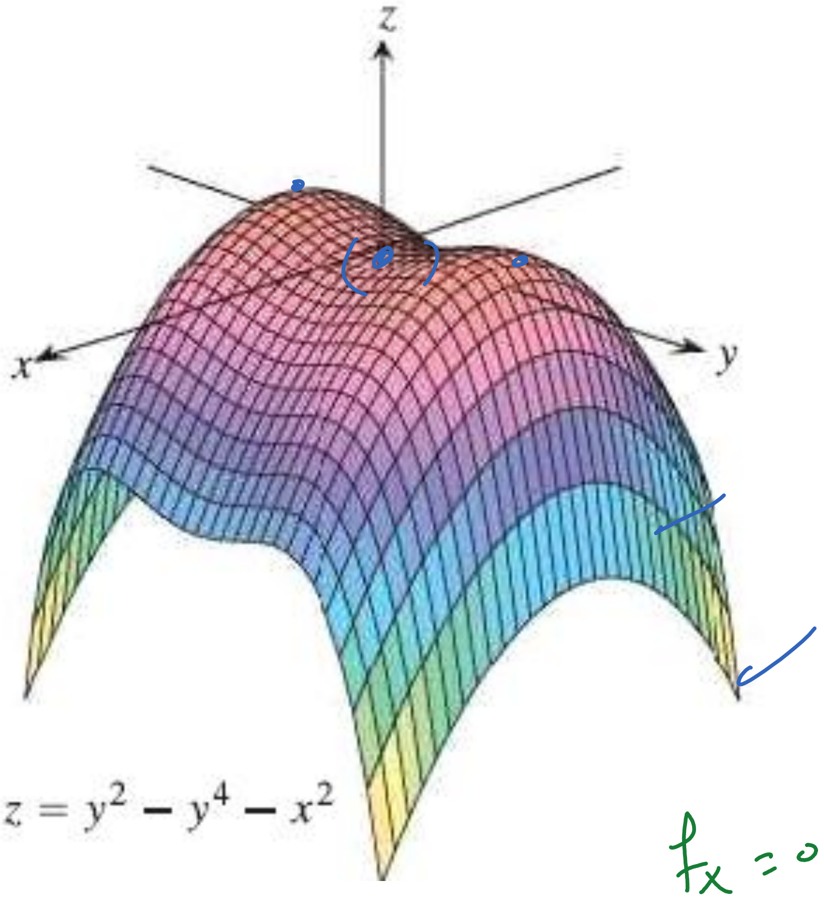
1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).



THEOREM 10—First Derivative Test for Local Extreme Values If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

DEFINITION An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

DEFINITION A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.42).



$f_x = 0$ —— ①

$f_y = 0$ —— ②

THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f. It is sometimes easier to remember it in determinant form,

$$\Delta(x,y) = \underbrace{f_{xx}f_{yy} - f_{xy}^2} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

$f_x = 0,\ f_y = 0 \Rightarrow \ldots$

Δ

- $\Delta > 0$
  - $f_{xx} > 0$ min.
  - $f_{xx} < 0$ max.
- $\Delta = 0$ test fails
- $\Delta < 0$ Saddle point

Ex. Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Sol. $f_x = y - 2x - 2 = 0 \Rightarrow \boxed{y - 2x = 2}$ --- ①

$f_y = x - 2y - 2 = 0 \Rightarrow \boxed{x - 2y = 2}$ --- ②

(Since f is diffble for all $(x,y) \Rightarrow f_x + f_y$ exists.)

$\dot{2}y - 4x = 4$

$x - 2y = 2$

Add $-3x = 6 \Rightarrow \boxed{x = -2} \xrightarrow{eq①} y + 4 = 2$, $\boxed{y = -2}$

the only critical point is $(-2,-2)$.

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2$

$= (-2)(-2) - (1)^2 = 3.$

$\therefore D(-2,-2) = 3 > 0$

$f_{xx}(-2,-2) = -2 < 0$

$\Rightarrow$ f has a local max. at $(-2,-2)$

the value of f at this point is

$f(-2,-2) = 8$ (check).

Example. Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Sol. $\boxed{f_x = -6x+6y} = 0 \Rightarrow \boxed{y = x}$ ---- ①

$f_y = \underbrace{6y - 6y^2 + 6x = 0} \Rightarrow \boxed{y - y^2 + x = 0}$ ②

① & ② $\Rightarrow$ $x - x^2 + x = 0 \Rightarrow 2x - x^2 = 0$

$x(2-x) = 0 \Rightarrow \boxed{x=0}$ or $\boxed{x=2}$

$x=0 \xrightarrow{eq ①} y=0$ $\boxed{(0,0)}$

$x=2 \xrightarrow{eq ①} y=2$ $\boxed{(2,2)}$

$f_{xx} = -6, \quad f_{xy} = 6, \quad f_{yy} = 6-12y.$

$\Delta(x,y) = f_{xx} f_{yy} - f_{xy}^2 = (-6)(6-12y) - 6^2$

$= -72 + 72y.$

$\Delta(0,0) = -72 < 0 \Rightarrow f$ has a saddle point at $(0,0)$.

$\left\{ \begin{array}{l} \Delta(2,2) = -72 + 72(2) = 72 > 0 \\ f_{xx}(2,2) = -6 < 0 \end{array} \right.$

$\Rightarrow$ f has a local max. at $(2,2)$ and its value is $f(2,2) = 8$ (check).

$D_f = \{(x,y) : x+y>0\}$

Q30) $f(x,y) = \ln(x+y) + x^2 - y$.

Sol. $f_x = \frac{1}{x+y} + 2x = 0$ - - - ①

$f_y = \frac{1}{x+y} - 1 = 0$ - - - ②

Eq① - Eq②: $2x+1=0 \Rightarrow \boxed{x = -\frac{1}{2}}$

From Eq② $\Rightarrow \frac{1}{y-\frac{1}{2}} = 1 \Rightarrow \boxed{y = 3/2}$

$\therefore$ the Critical point is $(-\frac{1}{2}, \frac{3}{2})$.

$f_{xx} = \frac{-1}{(x+y)^2} + 2$, $f_{yy} = \frac{-1}{(x+y)^2}$

$f_{xy} = \frac{-1}{(x+y)^2}$

$f_{xx}(-\frac{1}{2}, \frac{3}{2}) = 1$, $f_{yy}(-\frac{1}{2}, \frac{3}{2}) = -1$, $f_{xy}(-\frac{1}{2}, \frac{3}{2}) = -1$

$\therefore D(-\frac{1}{2}, \frac{3}{2}) = f_{xx} f_{yy} - f_{xy}^2 \Big|_{(-\frac{1}{2}, \frac{3}{2})}$

$= (1)(-1) - (-1)^2 = -2 < 0$

$\therefore$ f has a saddle point at $(-\frac{1}{2}, \frac{3}{2})$.

(23) $f(x,y) = y\sin x.$

$f_x = y\cos x = 0$ - - - - ①

$f_y = \boxed{\sin x = 0}$ - - - ②

Eq② $\Rightarrow$ $x = 0, \pm\pi, \pm 2\pi, \dots$

$\boxed{x = n\pi}$, $n = 0, \pm 1, \pm 2, \dots$

Eq① $\Rightarrow$ $y \underbrace{\cos(n\pi)}_{\neq 0} = 0$

$y(-1)^n = 0 \Rightarrow \boxed{y = 0}$

$\therefore$ the critical points are

$(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \dots$

$f_{xx} = -y\sin x$, $f_{yy} = 0$, $f_{xy} = \cos x$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2$

$= (-y\sin x)(0) - \cos^2 x = -\cos^2 x$

$D(n\pi, 0) = -\cos^2(n\pi) = -1 < 0$

$\therefore$ All $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \dots$

are Saddle points.

Q24) $f(x,y) = e^{2x} \cos y$.

$$f_x = \underbrace{\boxed{2e^{2x}}}_{\neq 0} \cos y = 0 \quad --- ① \Rightarrow \boxed{\cos y = 0}$$

$$f_y = \underbrace{\boxed{-e^{2x}}}_{\neq 0} \sin y = 0 \quad --- ② \Rightarrow \boxed{\sin y = 0}$$

⇒ there is no critical points

⇒ 〃 〃 〃 extreme values

or saddle points.

Q53) Find three numbers whose sum is 9 and whose sum of squares is a minimum.

Sol: Let $f(x,y,z) = x^2+y^2+z^2$

Given $x+y+z=9 \Rightarrow z = 9-x-y$

$$h(x,y) = f(x,y,z) = x^2+y^2+(9-x-y)^2$$

$$h_x = 2x + 2(9-x-y)(-1)$$

$$= 2x - 18 + 2x + 2y$$

$$\boxed{h_x = 4x+2y-18} = 0 \Rightarrow \boxed{2x+y=9} ①$$

$$h_y = 2y + 2(9-x-y)(-1) .$$

$$= 4y + 2x - 18 = 0 \Rightarrow \boxed{x+2y = 9} ... ②$$ ✓

eq① + eq②:

$$2x+y=9$$
$$-2x-4y=-18$$
$$-3y = -9 \Rightarrow \boxed{y=3} \Rightarrow 2x+3=9$$
$$\boxed{x=3}$$

$\boxed{z = 3}$

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$h_{xx} = 4, \quad h_{yy} = 4, \quad h_{xy} = 2$

$D(x,y) = h_{xx} h_{yy} - h_{xy}^2 = (4)(4) - 2^2$
$= 12 > 0$

$D(3,3) = 12 > 0$
$h_{xx}(3,3) = 4 > 0$ $\Rightarrow$ h has local min. at (3,3).

and its value $h(3,3) = 3^2 + 3^2 + (9-3-3)^2$
$= 27.$

In this section
triangle, rectangle, square

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

1. *List the interior points of R* where f may have local maxima and minima and evaluate f at these points. These are the critical points of f. $f_x = 0, \; f_y = 0$
2. *List the boundary points of R* where f has local maxima and minima and evaluate f at these points. We show how to do this shortly.
3. *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists made in Steps 1 and 2.

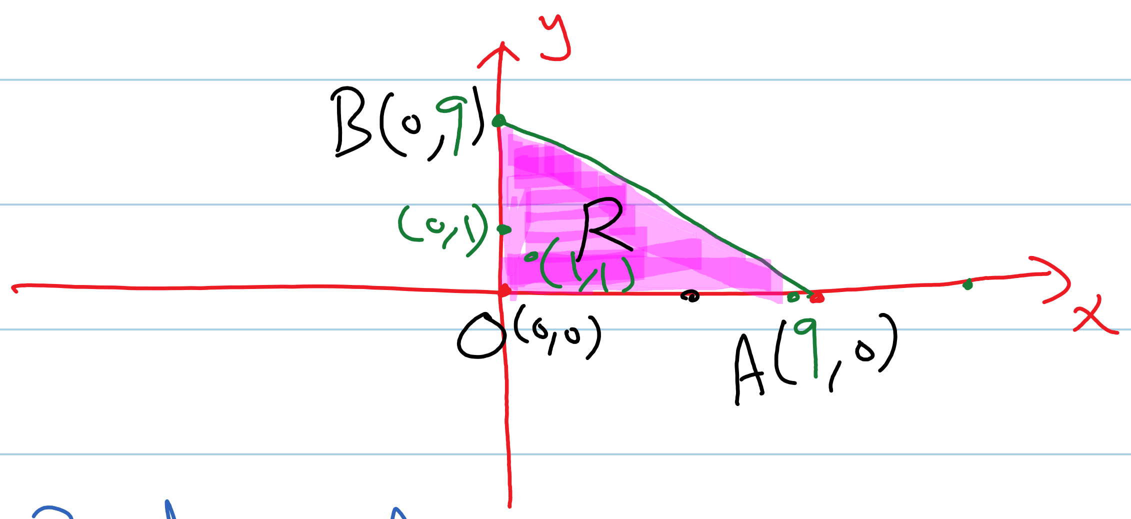
EXAMPLE 5 Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.

$x = 0 \Rightarrow y = 9$
$y = 0 \Rightarrow x = 9$

Sol. • Sketch the region.



• Interior Points: $f_x = 2 - 2x = 0 \Longrightarrow \boxed{x = 1}$

$f_y = 2 - 2y = 0 \Longrightarrow \boxed{y = 1}$

$\boxed{(x, y) = (1, 1)} \in$ Region.

Boundary points:

$\overline{OA}$: $\boxed{y=0}$, $0 \le x \le 9$.

$f(x,y) = f(x,0) = 2+2x-x^2$

$x=0 \Rightarrow \boxed{(0,0)}$. End points

$x=9 \Rightarrow \boxed{(9,0)}$

$f'(x,0) = 2-2x=0 \Rightarrow \boxed{x=1} \in [0,9]$

$\boxed{(1,0)}$

$\overline{OB}$: $\boxed{x=0}$, $0 \le y \le 9$

$f(0,y) = 2+2y-y^2$

مكرر $(0,0)$, $\boxed{(0,9)}$ End points

$f'(0,y) = 2-2y=0 \Rightarrow \boxed{y=1}$

$(0,1)$

$\overline{AB}$: $y = 9-x$, $0 \le x \le 9$

$x=0 \Rightarrow y=9$ $(0,9)$ مكرر

$x=9 \Rightarrow y=0$ $(9,0)$ مكرر

$f(x,y) = f(x,9-x) = 2+2x+2(9-x)-x^2$

Simplify. $-(9-x)^2$

$= -61+18x-2x^2$

$f'(x,9-x) = 18-4x=0 \Rightarrow x=9/2 \Rightarrow y=9-\frac{9}{2}=\frac{9}{2}$

$$\Rightarrow \boxed{\left(\frac{9}{2}, \frac{9}{2}\right)} \checkmark$$

∴ Summary.

	(x,y)	$f(x,y) = 2+2x+2y-x^2-y^2$
Interior point	$(1,1)$	$f(1,1) = 2+2+2-1-1 = \boxed{4}$
Boundary points	$(0,0)$	$f(0,0) = \boxed{2}$
	$(9,0)$	$f(9,0) = \boxed{-61}$
	$(0,9)$	$f(0,9) = \boxed{-61}$
	$(1,0)$	$f(1,0) = \boxed{3}$
	$(0,1)$	$f(0,1) = \boxed{3}$
	$(\frac{9}{2},\frac{9}{2})$	$f(\frac{9}{2},\frac{9}{2}) = \boxed{-\frac{41}{2}}$

the absolute max. is 4 occurs at (1,1)

the absolute min. is -61 occurs at (0,9) and (9,0)

14.8 Lagrange Multipliers

Constrained Maxima and Minima

EXAMPLE 1 Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin. $O(0,0,0)$

Sol. The problem is to find the min. value of the function

$$|\overrightarrow{OP}| = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

$$\boxed{d(x,y,z) = \sqrt{x^2+y^2+z^2}}$$ ✓

Subject to the constraint $\boxed{2x+y-z=5}$

First, find the min. of

$$f(x,y,z) = x^2+y^2+z^2$$

$$\boxed{h(x,y) = x^2+y^2+(2x+y-5)^2}$$

$$h_x = 2x+2(2x+y-5)(2) = 0$$

$$10x+4y = 20$$

$$\boxed{5x+2y=10} \cdots\cdots ①$$

$$h_y = 2y+2(2x+y-5)(1) = 0$$

$$4x+4y = 10$$

$$\boxed{2x+2y=5} \text{ — } ②$$

From ① & ② $\Rightarrow x = \frac{5}{3}, \quad y = \frac{5}{6}$ (check)

the critical value $\left(\frac{5}{3}, \frac{5}{6}\right)$

$$h_{xx} = 10, \quad h_{yy} = 4, \quad h_{xy} = 2$$

$$D(x,y) = h_{xx} h_{yy} - h_{xy}^2 = 40 - 4 = 36$$

$$\left.\begin{matrix} D\left(\frac{5}{3}, \frac{5}{6}\right) = 36 > 0 \\ h_{xx}\left(\frac{5}{3}, \frac{5}{6}\right) = 10 > 0 \end{matrix}\right\} \Rightarrow h \text{ is min. at } \left(\frac{5}{3}, \frac{5}{6}\right)$$

$$z = 2x + y - 5$$

$$= 2\left(\frac{5}{3}\right) + \frac{5}{6} - 5 = -\frac{5}{6}$$

closet point : $P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$.

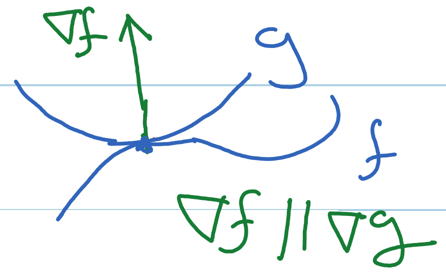
closet distance from P to O(0,0,0)

is $\sqrt{\frac{25}{9} + \frac{25}{36} + \frac{25}{36}} = \frac{\sqrt{150}}{6}$

$$= \frac{\sqrt{25(6)}}{6} \checkmark$$

$$= \frac{5\sqrt{6}}{6} = \frac{5}{\sqrt{6}}.$$

The Method of Lagrange Multipliers



The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

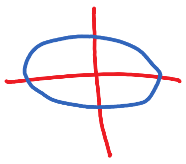
$$\boxed{\nabla f = \lambda \nabla g} \quad \text{and} \quad \boxed{g(x, y, z) = 0} \qquad (1)$$

For functions of two independent variables, the condition is similar, but without the variable z.

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.52)



$$\boxed{\frac{x^2}{8} + \frac{y^2}{2} = 1} \quad \text{Constraint .}$$

Sol: $f(x,y) = xy$, $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

Let $\nabla f = \lambda \nabla g$

$$f_x i + f_y j = \lambda(g_x i + g_y j)$$

$$y i + x j = \lambda\left(\frac{x}{4} i + y j\right)$$

$$\Rightarrow \begin{cases} y = \lambda \frac{x}{4} \text{ ---- ①} \\ x = \lambda y \text{ ---- ②} \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 \text{ ---- ③} \end{cases}$$

② into ①: $y = \frac{\lambda}{4}(\lambda y) \Rightarrow 4y - \lambda^2 y = 0$

$$y(4-\lambda^2)=0$$

$$\boxed{y=0} \quad \text{or} \quad \boxed{\lambda=\pm 2}$$

Case 1 $y=0 \xrightarrow{eq②} x=\lambda(0)=0$

$\boxed{(0,0)}$ reject since $\frac{0^2}{8}+\frac{0^2}{2}\neq 1$.

Case 2 $\boxed{\lambda=2} \xrightarrow{eq②} \boxed{x=2y}$

$$\xrightarrow{eq③} \frac{(2y)^2}{8}+\frac{y^2}{2}=1$$

$$y^2=1 \Rightarrow y=\pm 1$$

$y=1 \Rightarrow x=2$ $\boxed{(2,1)}$ ✓

$y=-1 \Rightarrow x=-2$ $\boxed{(-2,-1)}$ ✓

Case 3 $\boxed{\lambda=-2} \xrightarrow{eq②} \boxed{x=-2y}$

$$\xrightarrow{eq③} \frac{(-2y)^2}{8}+\frac{y^2}{2}=1$$

$$y^2=1 \Rightarrow \boxed{y=\pm 1}$$

$y=1 \Rightarrow x=-2$ $(-2,1)$ ✓

$y=-1 \Rightarrow x=2$ $(2,-1)$ ✓

(x,y)	$(2,1)$	$(-2,-1)$	$(-2,1)$	$(2,-1)$
$f(x,y)=xy$	2	2	-2	-2

abs. max. value = 2 occurs at

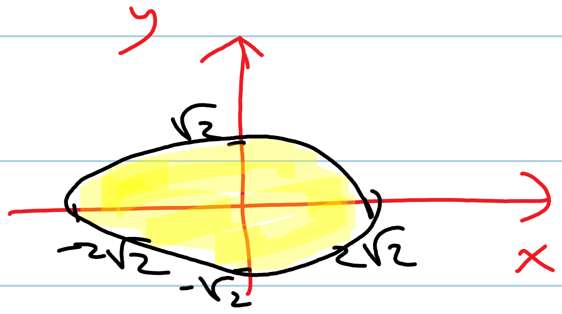
(2,1) and (−2,−1)

abs. min. value = −2 occurs at

(2,−1) and (−2,1).

Ex. Find the max. and min. of $f(x,y) = xy$

on $\frac{x^2}{8} + \frac{y^2}{2} \leq 1$



Sol. Interior Points:

$f_x = y = 0$ $\boxed{(0,0)} \in$ Region. $\frac{0^2}{8} + \frac{0^2}{2} \leq 1$ ✓

$f_y = x = 0$

Boundary Points $\frac{x^2}{8} + \frac{y^2}{2} = 1$ (Lagrange)

$\nabla f = \lambda \nabla g$ (see the last ex.).

(2,1), (−2,−1), (2,−1), (−2,1).

	Interior	Boundary			
(x,y)	$(0,0)$	$(2,1)$	$(-2,-1)$	$(2,-1)$	$(-2,1)$
$f(x,y)$	0	2	2	−2	−2

Abs. max = 2 occurs at (2,1) and (−2,−1)

Abs. min. = −2 " " (2,−1) and (−2,1).

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Ex. Find the max. and min. of $f(x,y,z) = x-2y+5z$ on the sphere $x^2+y^2+z^2 = 30$.

Sol. Let $g(x,y,z) = x^2+y^2+z^2-30=0$

& Let $\nabla f = \lambda \nabla g$

$$i-2j+5k = \lambda(2x\,i+2y\,j+2z\,k)$$

Notice that $\lambda \neq 0$

$$\Rightarrow \begin{cases} 2x\lambda = 1 & \text{—— ①} \Rightarrow \boxed{x = \frac{1}{2\lambda}} \\ 2y\lambda = -2 & \text{—— ②} \Rightarrow y = -\frac{1}{\lambda} \\ 2\lambda z = 5 & \text{—— ③} \Rightarrow z = \frac{5}{2\lambda} \\ x^2+y^2+z^2 = 30 & \text{—— ④} \end{cases}$$

put x, y, z into eq ④:

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = 30$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = 30$$

$$\frac{30}{4\lambda^2} = 30 \Rightarrow \boxed{\lambda = \pm\frac{1}{2}}$$

$\boxed{\lambda = \frac{1}{2}} \Rightarrow x = \frac{1}{2(\frac{1}{2})} = 1, \quad y = \frac{-1}{1/2} = -2$

$z = \frac{5}{2(\frac{1}{2})} = 5$

$P_1(1,-2,5).$

$\boxed{\lambda = -\frac{1}{2}} \Rightarrow P_2(-1,2,-5).$

$$f(P_1) = f(1,-2,5)$$
$$= 1 - 2(-2) + 5(5) = 30$$

$$f(P_2) = f(-1,2,-5) = -1 - 2(2) + 5(-5)$$
$$= -30$$

Abs. max $= 30$ occurs at $(1,-2,5)$.

" min. $= -30$ " " $(-1,2,-5)$.

Ex. Find the extreme values of $f(x,y,z) = xy + z^2$ subject to the constraint: $x^2+y^2+z^2=1$

Sol. Let $g(x,y,z) = x^2+y^2+z^2-1=0$.

Let $\nabla f = \lambda \nabla g$

$$y\,i + x\,j + 2z\,k = \lambda(2x\,i + 2y\,j + 2z\,k)$$

$$\begin{cases} y = 2x\lambda & \text{——} ① \\ x = 2y\lambda & \text{——} ② \\ 2z = 2z\lambda & \text{——} ③ \\ x^2+y^2+z^2=1 & \text{——} ④ \end{cases}$$

put ② into ① : $y = 2(2y\lambda)\lambda$

$$y - 4y\lambda^2 = 0 \Rightarrow y(1-4\lambda^2) = 0$$

$\boxed{y=0}$ or $\boxed{\lambda = \pm\frac{1}{2}}$

Case 1 $\boxed{y=0}$ $\xrightarrow{eq②}$ $\boxed{x=0}$

$x=0,\ y=0 \xrightarrow{④} z^2=1$

$z=\pm 1$

$\boxed{P_1(0,0,1)}$, $\boxed{P_2(0,0,-1)}$

Case 2 $\boxed{\lambda=-\frac{1}{2}}$ eq① or ② $\Rightarrow$ $\boxed{y=-x}$

eq③ $2z=-z \Rightarrow 3z=0$

$\boxed{z=0}$ ✓

eq④ $x^2+y^2+z^2=1$

$2x^2=1 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$

$x=\frac{1}{\sqrt{2}}, y=-\frac{1}{\sqrt{2}}$ $P_3\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

$x=-\frac{1}{\sqrt{2}} \Rightarrow y=\frac{1}{\sqrt{2}}$ $P_4\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

Case 3 $\boxed{\lambda=\frac{1}{2}}$ eq① or ② $\Rightarrow$ $\boxed{y=x}$

eq③ $2z=z \Rightarrow$ $\boxed{z=0}$

eq④ $x^2+y^2+z^2=1$

$2x^2=1 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$

$x=\frac{1}{\sqrt{2}} \Rightarrow y=\frac{1}{\sqrt{2}}$ $P_5\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$x=\frac{-1}{\sqrt{2}}, y=\frac{-1}{\sqrt{2}}$ $P_6\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$.

Summary.

(x,y,z)	$f(x,y,z) = xy+z^2$
$(0,0,1)$	1
$(0,0,-1)$	1
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$	$+\frac{1}{2}$
$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$	$+\frac{1}{2}$
$\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$	$-\frac{1}{2}$
$\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$	$-\frac{1}{2}$

Absol. max $= 1$ occurs at $(0,0,\pm 1)$

Abs. min. $= -\frac{1}{2}$ " " $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

and $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

Review (8.1, 8.2, 8.3, 8.4)

15

MULTIPLE INTEGRALS

15.1 Double and Iterated Integrals over Rectangles

Double Integrals

The **double integral** of f over R, written as

dxdy or dydx

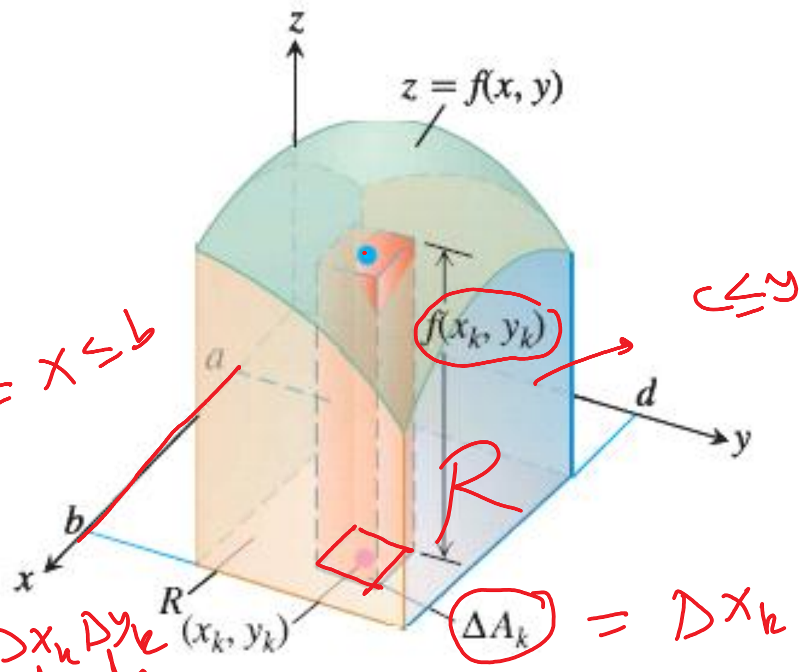
$$\iint_R f(x, y)\, dA \quad \text{or} \quad \iint_R f(x, y)\, dx\, dy.$$

Region in xy-plane

Double Integrals as Volumes

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x, y)\, dA,$$

where $\Delta A_k \to 0$ as $n \to \infty$.



$a \le x \le b$

$c \le y \le d$

$\Delta A_k = \Delta x_k \cdot \Delta y_k$

$f(x_k, y_k)\, \Delta x_k\, \Delta y_k$

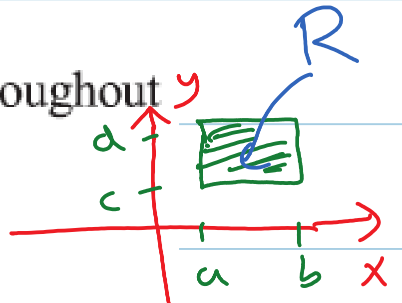
$dx\, dy$

FIGURE 15.2 Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R.

If $f(x,y) = 1$

$$\iint_R dA = \text{Area of } R.$$

Fubini's Theorem for Calculating Double Integrals



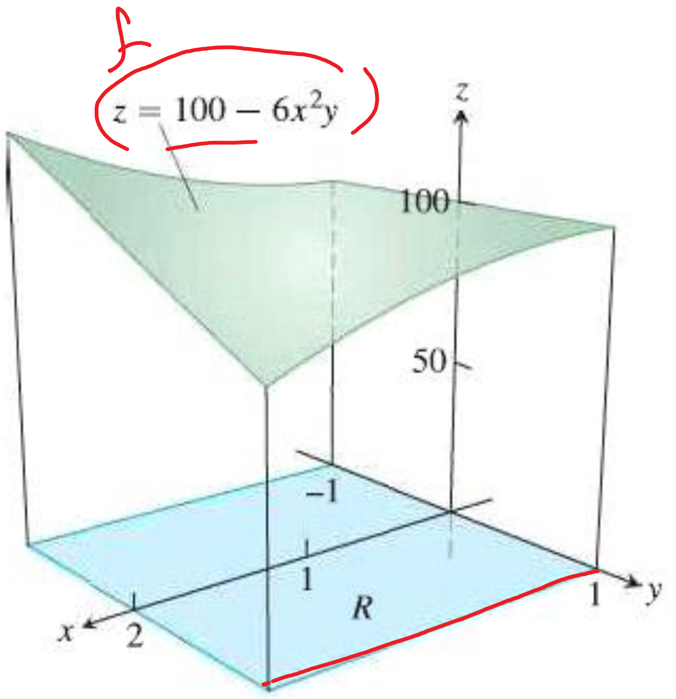
THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b, c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{k(y)}\,dy = \int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{h(x)}\,dx. \checkmark$$

$\int_c^d k(y)dy$ $\quad$ $\int_a^b h(x)dx$ ✓

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R: \; 0 \le x \le 2, \quad -1 \le y \le 1.$$



$$I = \iint_R f(x,y)\,dA$$

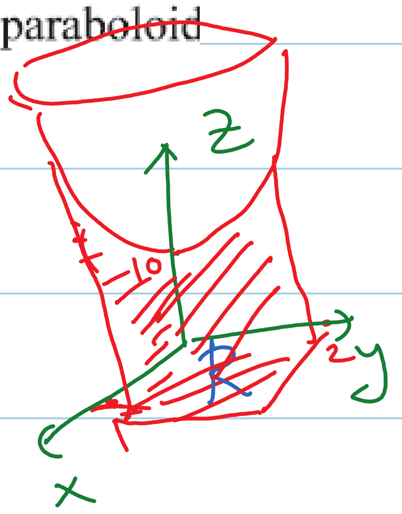
$$= \int_0^2 \underbrace{\int_{-1}^{1} (100 - 6x^2y)\,dy}\,dx \checkmark$$

$$= \int_0^2 100y - \frac{6x^2y^2}{2}\Bigg|_{y=-1}^{y=1} dx$$

$$= \int_0^2 \left[(100 - 3x^2) - (-100 - 3x^2)\right] dx$$

$$= \int_0^2 200\,dx = 200x\Big|_0^2 = 400$$

EXAMPLE 2 Find the volume of the region bounded above by the ellipitical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1, 0 \le y \le 2$.



$$\text{Volume} = \iint_R \overset{z}{f(x,y)}\,dA$$

$$= \int_0^2\int_0^1 (10 + x^2 + 3y^2)\,dx\,dy$$

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$$= \int_0^2 \boxed{\int_0^1 \left(10 + x^2 + 3y^2\right) dx}\, dy$$

$$= \int_0^2 \left(10x + \frac{x^3}{3} + 3y^2 x\right)\Bigg|_{x=0}^{x=1} dy$$

$$= \int_0^2 \left(10 + \frac{1}{3} + 3y^2\right) - (0) \quad dy$$

$$= \int_0^2 \left(\frac{31}{3} + 3y^2\right) dy \qquad (\text{Cal } 1)$$

$$= \frac{31}{3}y + y^3 \Big|_0^2 = \left(\frac{31}{3}(2) + 8\right) - (0)$$

$$= \frac{62}{3} + 8 = \frac{86}{3}$$

15. $I = \iint_R xy\cos y\, dA, \qquad R: \quad -1 \leq x \leq 1, \quad 0 \leq y \leq \pi$

$$I = \int_0^{\pi} \boxed{\int_{-1}^{1} xy\cos y \; dx}\, dy$$

$$= \int_0^{\pi} \frac{x^2}{2} y\cos y \Bigg|_{x=-1}^{x=1} dy$$

$$= \int_0^{\pi} \left(\frac{1}{2} y\cos y - \frac{1}{2} y\cos y\right) dy$$

$$= \int_0^{\pi} 0 \, dy = 0$$

18. $I = \iint_R xye^{xy^2}\,dA, \qquad R: \quad 0 \le x \le 2, \quad 0 \le y \le 1$

$$I = \underbrace{\int_0^2 \int_0^1 xy\, e^{\boxed{xy^2}}\, dy\, dx}$$

$u = xy^2$

$\frac{du}{dy} = 2xy \Rightarrow \boxed{\frac{1}{2}du = xy\,dy}$

$y = 0 \Rightarrow u = 0$

$y = 1 \Rightarrow u = x(1)^2 = x$

$$= \int_0^2 \left(\int_0^x \frac{1}{2} e^u\, du \right) dx$$

$$= \int_0^2 \frac{1}{2} e^u \Big|_0^x \, dx$$

$$= \int_0^2 \frac{1}{2}(e^x - 1)\, dx$$

$$= \frac{1}{2}(e^x - x)\Big|_0^2 = \frac{1}{2}(e^2 - 2) - \frac{1}{2}(1 - 0)$$

$$= \frac{1}{2}e - \frac{3}{2} = \frac{e-3}{2}.$$

19. $I = \iint_R \frac{xy^3}{x^2+1}\,dA, \qquad R: \quad 0 \le x \le 1, \quad 0 \le y \le \cancel{2}4$

$$I = \int_0^1 \boxed{\int_0^4 \frac{x y^3}{x^2+1}\,dy}\,dx$$

$$= \int_0^1 \frac{x}{x^2+1} \cdot \frac{y^4}{4} \Bigg|_{y=0}^{y=4} dx$$

$$= \int_0^1 \frac{x}{x^2+1} \left(\frac{4^4}{4} - \frac{0^4}{4} \right) dx$$

$$= 32 \int_0^1 \frac{2x}{x^2+1}\,dx \qquad \text{(Cal1)}$$

$$= 32 \ln(x^2+1) \Big|_0^1$$

$$= 32 (\ln 2 - \ln 1) = 32 \ln 2$$

27. Find the volume of the region bounded above by the surface $z = 2 \sin x \cos y$ and below by the rectangle $R\colon 0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

Sol. $\text{Volume} = \iint_R z\,dA$

$$= \int_0^{\frac{\pi}{4}} \boxed{\int_0^{\pi/2} 2 \sin x \cos y \, dx}\, dy$$

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$$= \int_0^{\pi/4} -2\cos x \cos y \Big|_{x=0}^{x=\frac{\pi}{2}} dy$$

$$= \int_0^{\pi/4} \left(-2\overset{0}{\cos\frac{\pi}{2}}\cos y + 2\overset{1}{\cos 0}\cos y\right) dy$$

$$= \int_0^{\pi/4} 2\cos y\, dy = 2\sin y \Big|_0^{\pi/4}$$

$$= 2\sin\frac{\pi}{4} - 2\sin 0$$

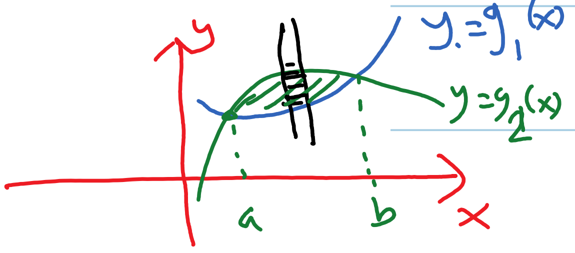
$$= 2\frac{\sqrt{2}}{2} = \sqrt{2}.$$

15.2 Double Integrals over General Regions

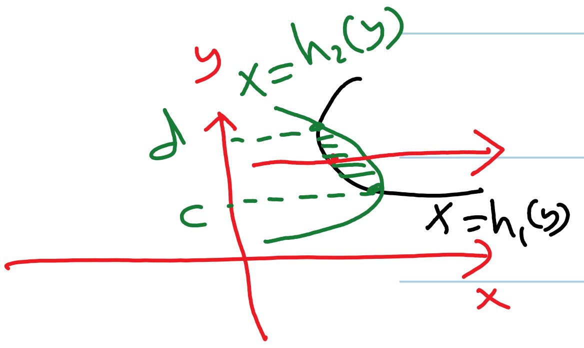
Double Integrals over Bounded, Nonrectangular Regions

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$



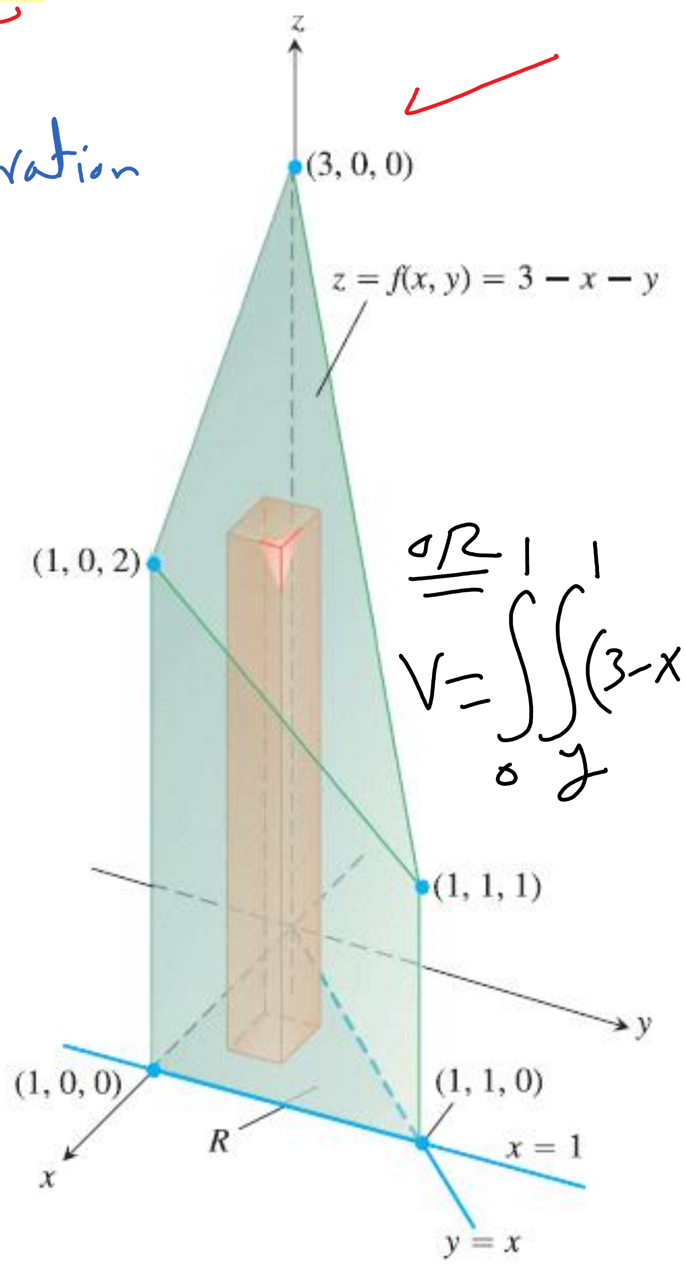
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$



منشور

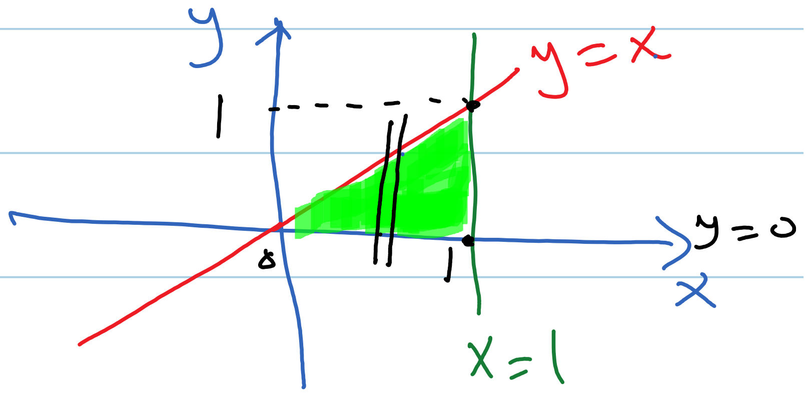
EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$



Sol. Sketch the region of integration

R: x-axis, $y = x$, $x = 1$



or R

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy$$

$$\text{Volume} = \iint_R f(x, y)\, dA$$

$$= \int_0^1 \int_0^x (3 - x - y)\, dy\, dx$$

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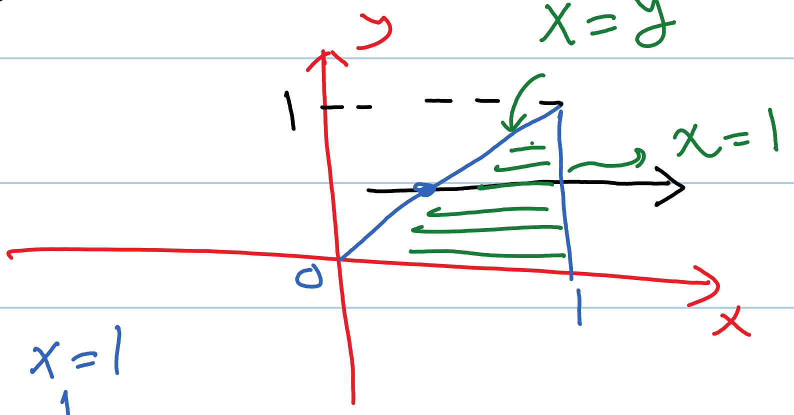
$$=\int_0^1 \left((3-x)y-\frac{y^2}{2}\right)\Big|_{y=0}^{y=x}\,dx$$

$$=\int_0^1\left[\left((3-x)x-\frac{x^2}{2}\right)-(0-0)\right]dx$$

$$=\int_0^1\left(3x-\frac{3}{2}x^2\right)dx=\left(\frac{3x^2}{2}-\frac{x^3}{2}\right)\Big|_0^1$$

$$=\frac{3}{2}-\frac{1}{2}=1$$

OR $$V=\int_0^1\int_y^1(3-x-y)\,\boxed{dx}\,dy$$



$$=\int_0^1\left(3x-\frac{x^2}{2}-yx\right)\Big|_{x=y}^{x=1}\,dy$$

$$=\int_0^1\left[\left(3-\frac{1}{2}-y\right)-\left(3y-\frac{y^2}{2}-y^2\right)\right]dy$$

$$=\int_0^1\left(\frac{5}{2}-4y+\frac{3}{2}y^2\right)dy$$

$$=\left(\frac{5}{2}y-2y^2+\frac{1}{2}y^3\right)\Big|_0^1=\frac{5}{2}-2+\frac{1}{2}$$

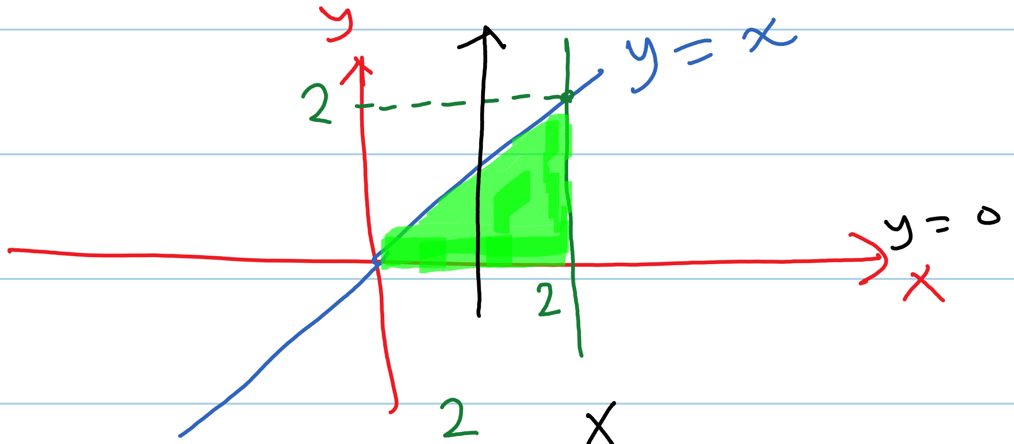
$$=1$$

EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $X = 2$

Sol.



$$\iint_R \frac{\sin x}{x}\, dA = \int_0^2 \underbrace{\int_0^x \frac{\sin x}{x}\, dy}\, dx$$

$$= \int_0^2 \frac{\sin x}{x} \cdot y \Big|_{y=0}^{y=x} dx$$

$$= \int_0^2 \left(\frac{\sin x}{x} \cdot x - 0\right) dx$$

$$= \int_0^2 \sin x\, dx = -\cos x \Big|_0^2$$

$$= 1 - \cos 2 .$$

Ex. Evaluate the integral

$$\int_0^3\int_{\sqrt{\frac{x}{3}}}^{1} e^{y^3}\,dy\,dx$$

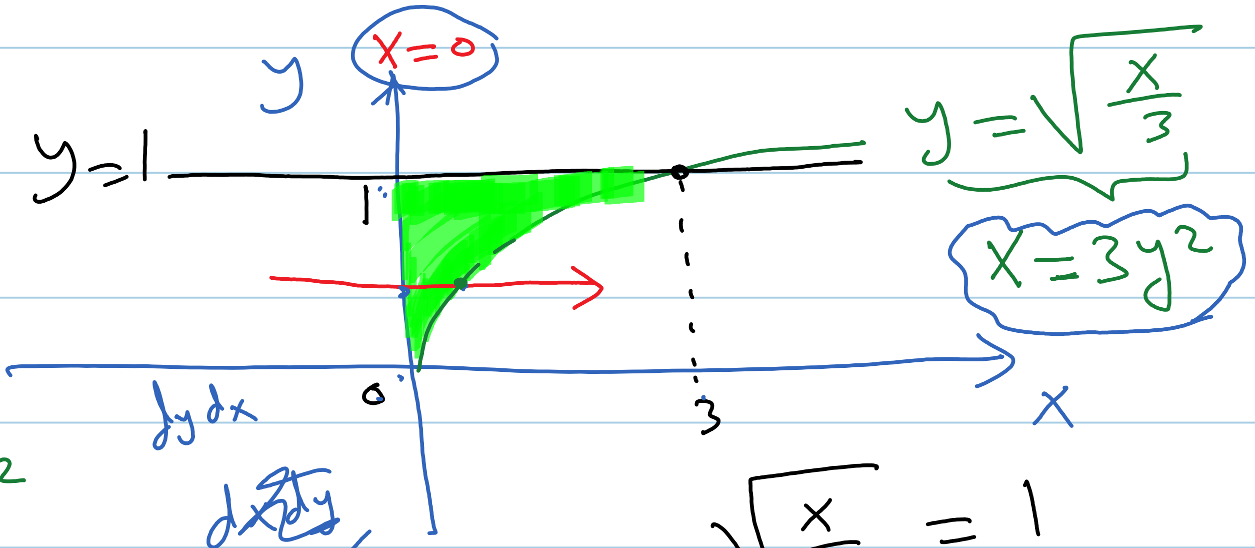
Sol. . Sketch the region of integration.

. Reverse the order of " .

. Evaluate.

Now, $\underbrace{0\le x\le 3}$, $\sqrt{\frac{x}{3}}\le y\le 1$

$y=\sqrt{\frac{x}{3}}$, $y=1$



dy dx

dx dy

$\sqrt{\frac{x}{3}}=1$

$\boxed{x=3}$

$$I=\int_0^1\int_0^{3y^2} e^{y^3}\,\underbrace{dx}\,dy$$

$$=\int_0^1 xe^{y^3}\Big|_{x=0}^{x=3y^2}dy=\int_0^1 3y^2e^{y^3}\,dy$$

$u=y^3$

$du=3y^2dy$

$y=0\Rightarrow u=0$

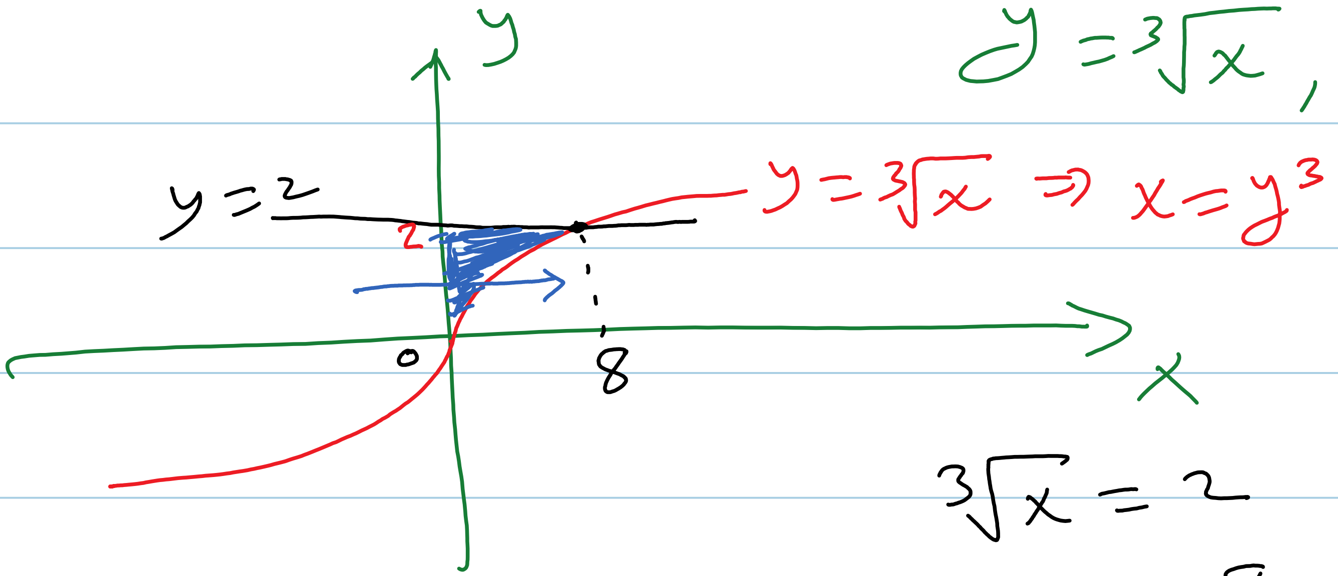
$y=1\Rightarrow u=1$

$$=\int_0^1 e^u\,du=e^u\Big|_0^1=e-1$$

54. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$

$0 \le x \le 8$

$\underbrace{\sqrt[3]{x} \le y \le 2}_{y=\sqrt[3]{x}\ ,\ y=2}$



$\sqrt[3]{x} = 2$

$x = 8$

$$I = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy$$

$$= \int_0^2 \frac{x}{y^4+1}\Big|_{x=0}^{x=y^3} dy = \frac{1}{4}\int_0^2 \frac{4y^3}{y^4+1}\, dy$$

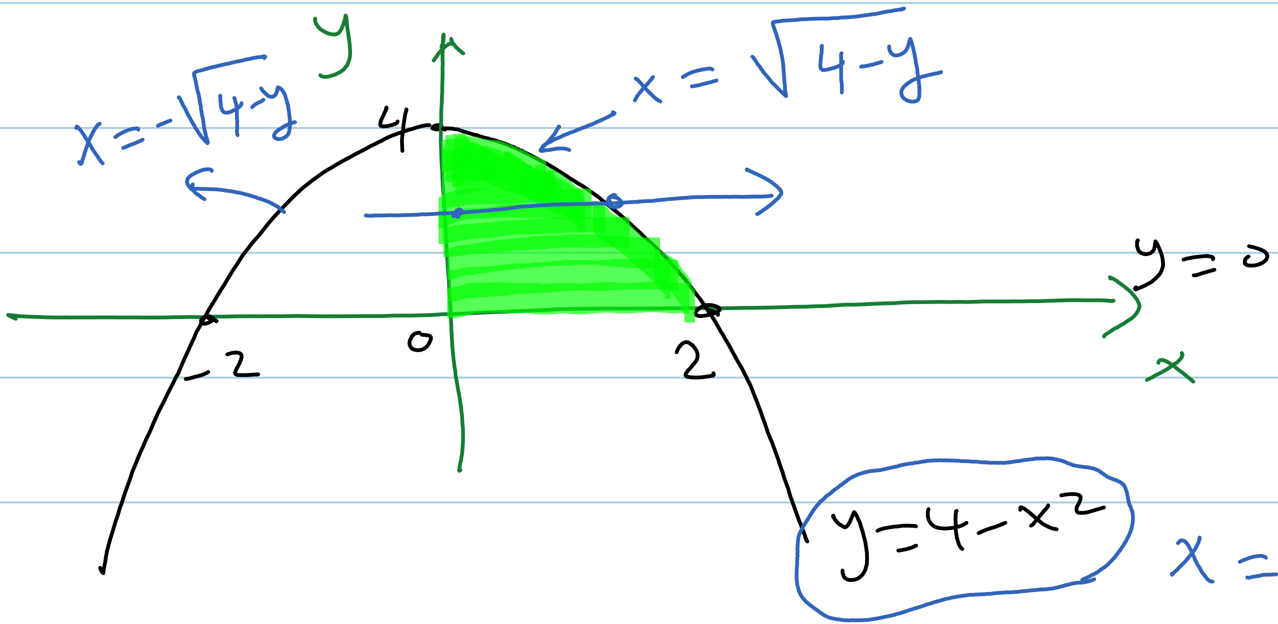
$$= \frac{1}{4} \text{Ln}(y^4+1)\Big|_0^2$$

$$= \frac{1}{4} \text{Ln} 17 .$$

50. $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx$

$0 \leq y \leq 4-x^2 \quad , \quad 0 \leq x \leq 2$

$y=0 , \quad y=4-x^2$



$x=\sqrt{4-y}$

$$I = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$$

$$= \int_0^4 \frac{x^2}{2} \frac{e^{2y}}{4-y} \Big|_{x=0}^{x=\sqrt{4-y}} dy$$

$$= \frac{1}{2}\int_0^4 \left(\sqrt{4-y}\right)^2 \frac{e^{2y}}{4-y}\, dy$$

$$= \frac{1}{2}\int_0^4 e^{2y}\, dy = \frac{1}{4} e^{2y}\Big|_0^4$$

$$= \frac{e^8-1}{4} .$$

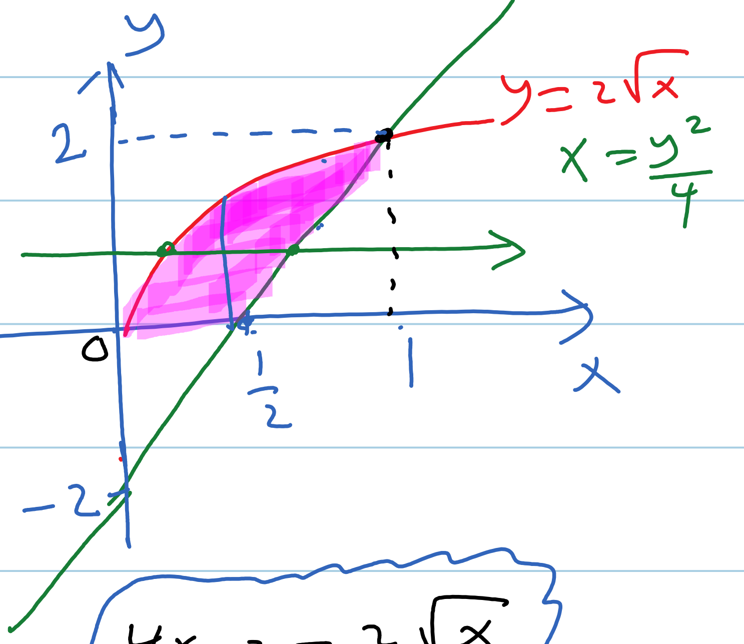
EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

$y = 4x - 2$

$x = \frac{y+2}{4}$

Sol. $f(x,y) = 16 - x^2 - y^2$

$$V = \int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} (16 - x^2 - y^2)\,dx\,dy$$



$$\boxed{4x - 2 = 2\sqrt{x}}$$

$$2x - 1 = \sqrt{x}$$

$$4x^2 - 4x + 1 = x$$

$$4x^2 - 5x + 1 = 0$$

$$(4x - 1)(x - 1) = 0$$

$$\underbrace{x = \tfrac{1}{4}}_{\text{reject}},\quad x = 1$$

OR

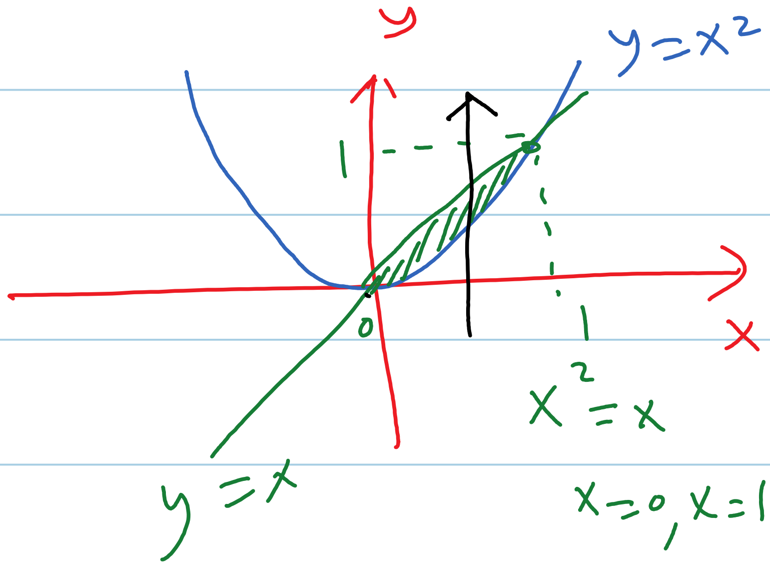
$$V = \int_0^{\frac{1}{2}} \int_0^{2\sqrt{x}} (16 - x^2 - y^2)\,dy\,dx + \int_{\frac{1}{2}}^{1} \int_{4x-2}^{2\sqrt{x}} (16 - x^2 - y^2)\,dy\,dx$$

15.3 Area by Double Integration

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R 1\, dA.$$

EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



$x^2 = x$

$x=0, x=1$

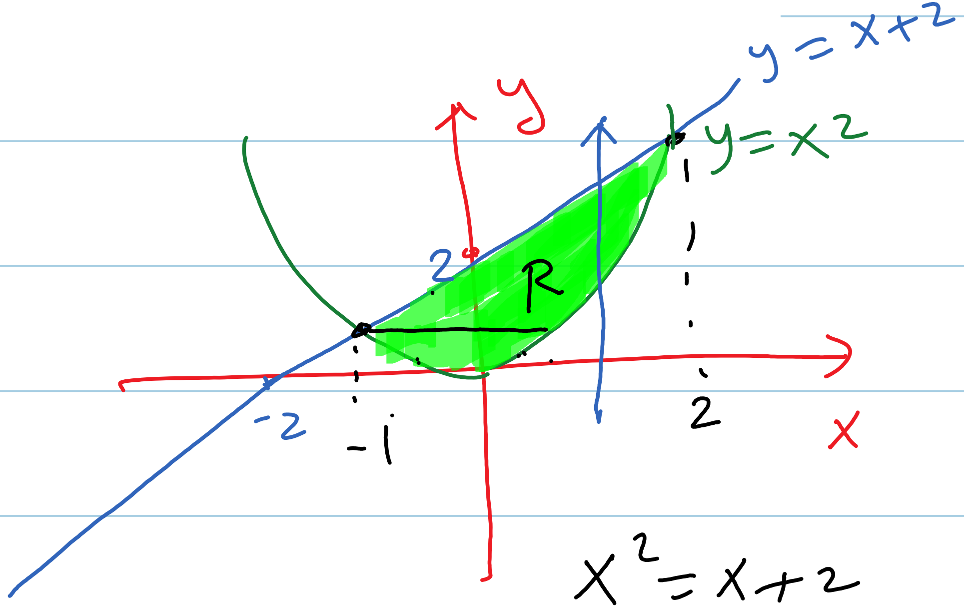
$$\text{Area} = \iint_R dA$$

$$= \int_0^1 \int_{x^2}^{x} 1\, dy\, dx$$

$$= \int_0^1 y \Big|_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2)\, dx \qquad \underline{\text{Call}}$$

$$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.



$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, \; x = -1$

$$\text{Area} = \iint_R dA$$

$$= \int_{-1}^{2} \int_{x^2}^{x+2} 1\, dy\, dx$$

$$= \int_{-1}^{2} (x + 2 - x^2)\, dx = \cdots\cdots$$

$$\iint_R 1\,dA = \text{Area}(R)$$

$$\iint_R f\,dA = \text{volume}$$

Average Value

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R}\iint_R f\,dA. = \frac{\text{Volume}}{\text{Area}}$$

EXAMPLE 3 Find the average value of $f(x, y) = x\cos xy$ over the rectangle $R: 0 \le x \le \pi,\ 0 \le y \le 1$.

Sol. Area $= \iint_R dA = \int_0^{\pi}\int_0^1 dy\,dx = \int_0^{\pi} y\Big|_{y=0}^{y=1} dx = \int_0^{\pi} 1\,dx = x\Big|_0^{\pi} = \pi$

$$\text{Area} = (\text{length})(\text{width}) = (\pi - 0)(1 - 0) = \pi.$$

$$\text{Volume} = \iint_R f(x,y)\,dA = \int_0^{\pi}\int_0^1 x\cos(xy)\,dy\,dx$$

$$= \int_0^{\pi} \frac{x\sin(xy)}{x}\Big|_{y=0}^{y=1} dx$$

$$= \int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$= 2$$

∴ Average value of f over R is

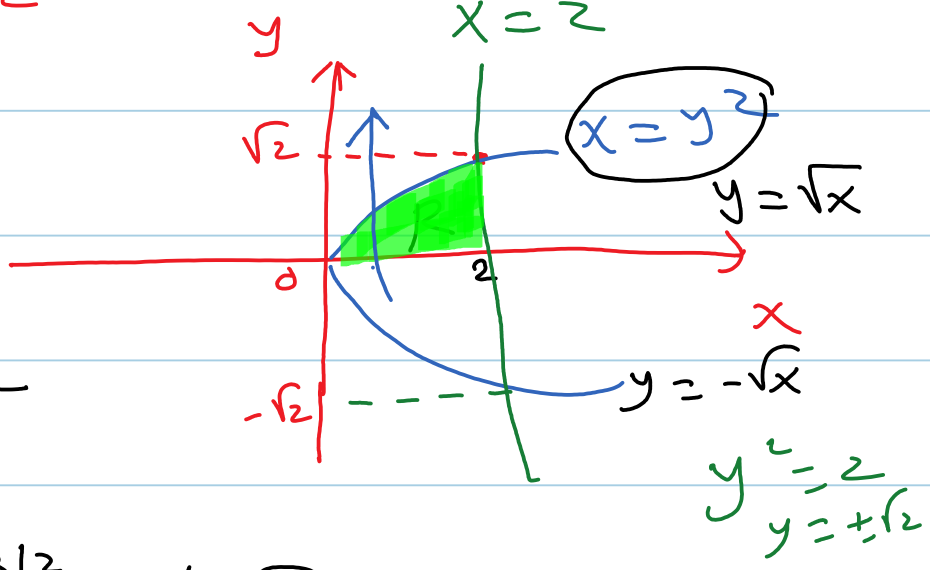
$$= \frac{1}{A}\iint_R f(x,y)\,dA = \frac{1}{\pi}(2) = \frac{2}{\pi}.$$

Ex. Find the average value of $f(x,y) = 12y^3 e^{x^3}$ over the region

R: $y^2 \le x \le 2, \quad 0 \le y \le \sqrt{2}$.

$x = y^2, \ x = 2$

Sol.



· Area $= \int_0^2 \int_0^{\sqrt{x}} dy\, dx$

$= \int_0^2 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_0^2$

$= \frac{2}{3}(2)^{3/2} = \frac{4\sqrt{2}}{3}$

· Volume $= \int_0^2 \underbrace{\int_0^{\sqrt{x}} 12y^3 e^{x^3} dy\, dx}$

$= \int_0^2 3y^4 e^{x^3} \Big|_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^2 3x^2 e^{x^3} dx$

$u = x^3$
$du = 3x^2 dx$

$x = 0 \Rightarrow u = 0$
$x = 2 \Rightarrow u = 8$

$= \int_0^8 e^u du$

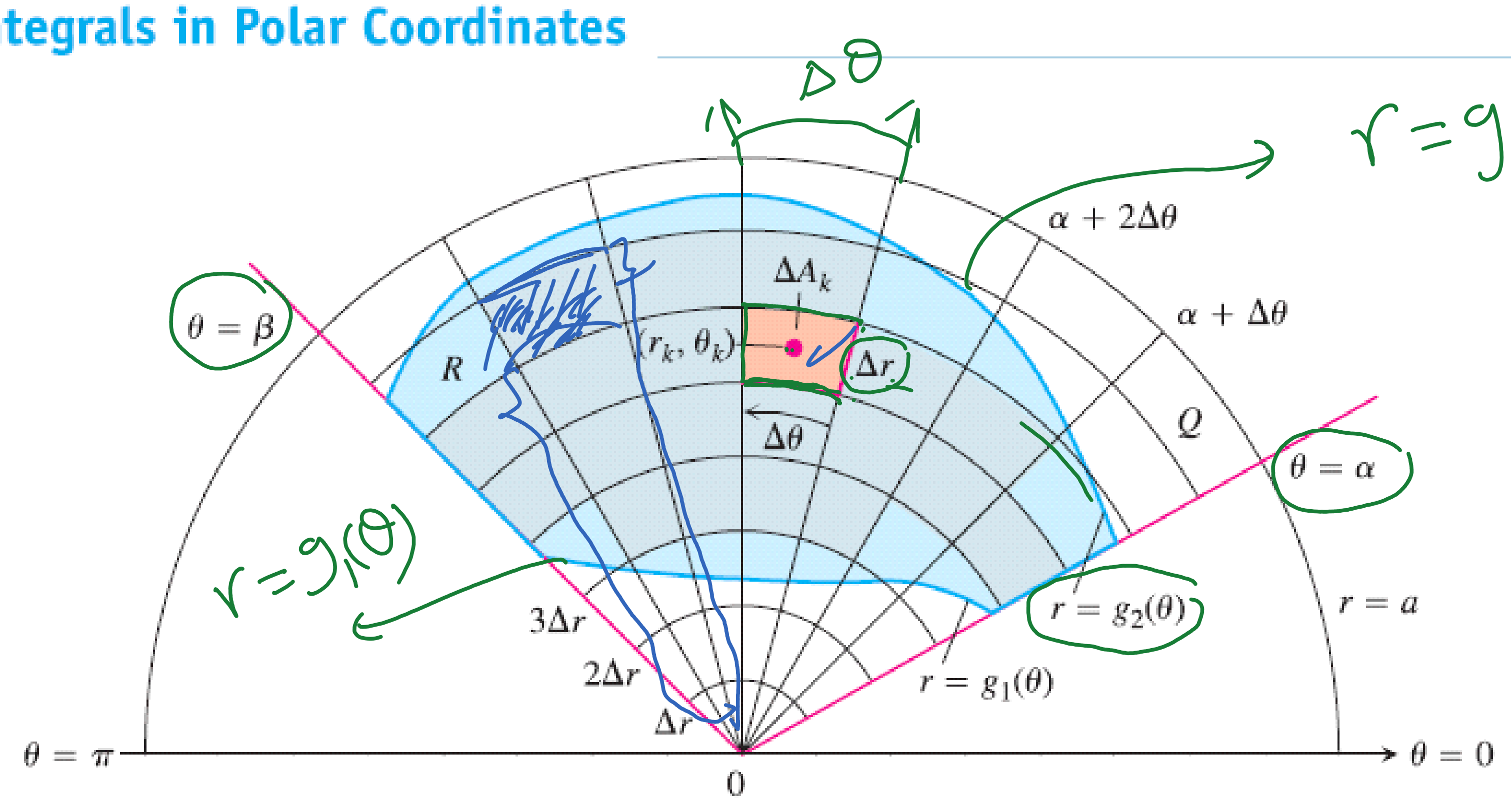
$= e^u \Big|_0^8 = e^8 - 1$

∴ Average value of f over R is

$$\frac{1}{\text{Area}} \iint_R f(x,y)\, dA = \frac{3}{4\sqrt{2}} (e^8 - 1).$$

15.4 Double Integrals in Polar Form

Integrals in Polar Coordinates



$r = g_2(\theta)$

$\alpha \leq \theta \leq \beta$

$DA = \Delta r \, \Delta\theta$

FIGURE 15.21 The region R: $g_1(\theta) \leq r \leq g_2(\theta)$, $\alpha \leq \theta \leq \beta$, is contained in the fan-shaped region Q: $0 \leq r \leq a$, $\alpha \leq \theta \leq \beta$. The partition of Q by circular arcs and rays induces a partition of R.

$$\theta = \alpha, \qquad \theta = \alpha + \Delta\theta, \qquad \theta = \alpha + 2\Delta\theta, \qquad \ldots, \qquad \theta = \alpha + m'\Delta\theta = \beta,$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into small patches called "polar rectangles."

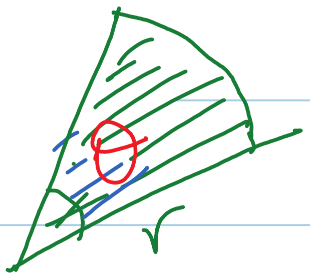
$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, \Delta A_k. \qquad \Delta r_k \Delta\theta_k$$

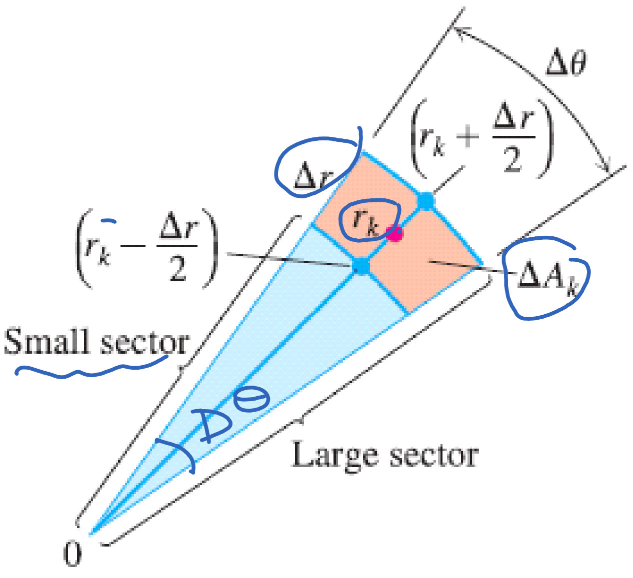
$$\lim_{n\to\infty} S_n = \boxed{\iint_R f(r, \theta)\, dA.}$$

$dA = r\, dr\, d\theta$

The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$A = \frac{1}{2}\theta \cdot r^2,$$





Inner radius: $\frac{1}{2}\left(r_k-\frac{\Delta r}{2}\right)^2\Delta\theta$

Outer radius: $\frac{1}{2}\left(r_k+\frac{\Delta r}{2}\right)^2\Delta\theta.$

$\frac{1}{2}\theta r^2$

$$\Delta A_k = \text{area of large sector} - \text{area of small sector}$$

$$= \boxed{\frac{\Delta\theta}{2}}\left[\left(r_k+\frac{\Delta r}{2}\right)^2-\left(r_k-\frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{\cancel{2}}(\cancel{2}r_k\,\Delta r) = \underset{r}{r_k}\,\underset{dr}{\Delta r}\,\underset{d\theta}{\Delta\theta}.$$

As $n\to\infty$ and the values of Δr and $\Delta\theta$ approach zero, these sums converge to the double integral

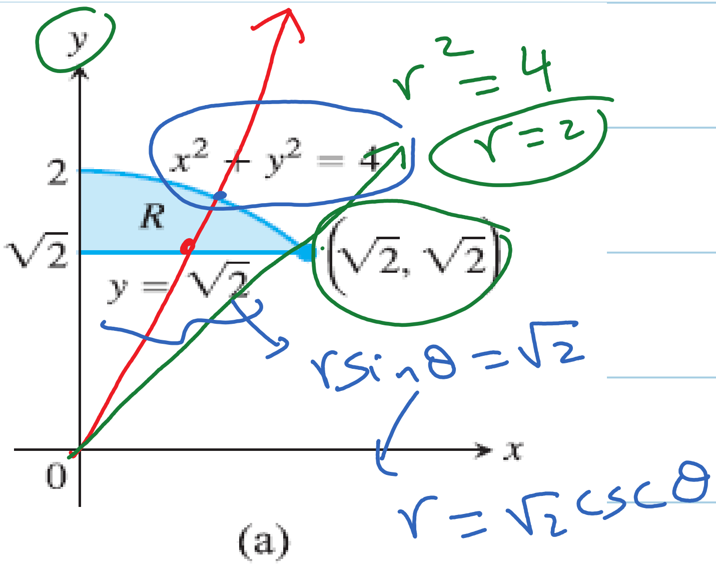
$dA = dx\,dy$ or $dy\,dx$

$$\lim_{n\to\infty} S_n = \iint_R f(r,\theta)\,\boxed{r\,dr\,d\theta}.$$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$

Finding Limits of Integration

1. *Sketch*. Sketch the region and label the bounding curves (Figure 15.23a).
2. *Find the r-limits of integration*. Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis (Figure 15.23b).
3. *Find the θ-limits of integration*. Find the smallest and largest θ-values that bound R. These are the θ-limits of integration (Figure 15.23c). The polar iterated integral is

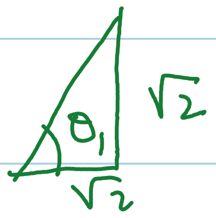


(a)

$r^2=4$
$r=2$
$r\sin\theta=\sqrt{2}$
$r=\sqrt{2}\csc\theta$

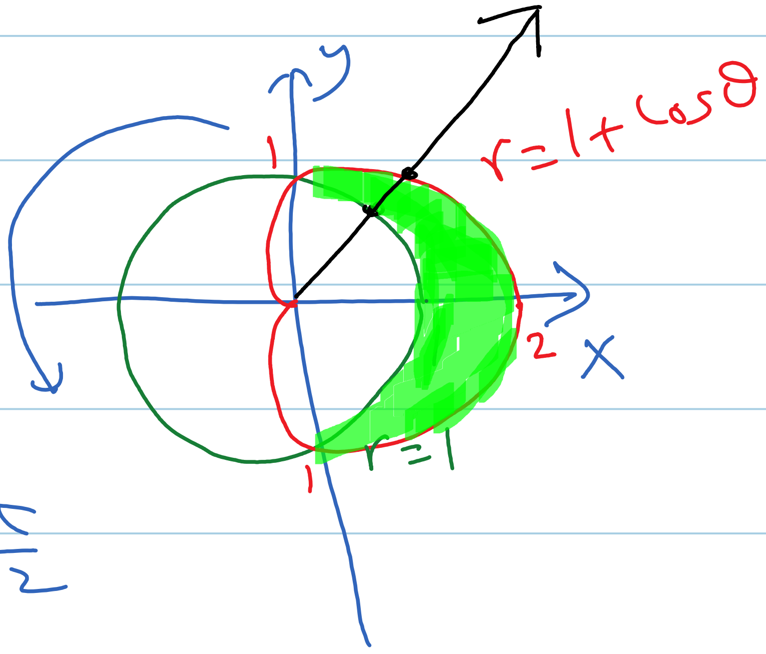
$$\iint_R f(x,y)\,dA$$

$$=\int_{\pi/4}^{\pi/2}\int_{\sqrt{2}\csc\theta}^{2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



$\tan\theta_1 = \frac{\sqrt{2}}{\sqrt{2}} = 1 \Rightarrow \theta = \frac{\pi}{4}$

EXAMPLE 1 Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $\underbrace{r = 1 + \cos\theta}$ and outside the circle $r = 1$.



$$1 \le r \le 1+\cos\theta$$

θ: $1+\cos\theta = 1$

$\cos\theta = 0$

$\theta = \pm\frac{\pi}{2}$

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\iint_R f(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint \boxed{r\,dr\,d\theta.}$$

$$\text{Area} = \iint_R dA$$

Changing Cartesian Integrals into Polar Integrals

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta,$$

EXAMPLE 3 Evaluate

$$I = \iint_R e^{x^2+y^2}\,\boxed{dy\,dx,}$$

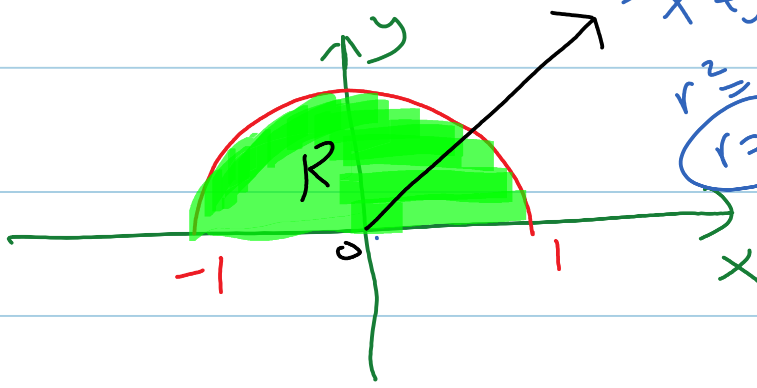
$$y^2 = 1 - x^2 \Rightarrow x^2+y^2=1$$

where R is the semicircular region bounded by the x-axis and the curve $\boxed{y = \sqrt{1 - x^2}}$

$\rightarrow x^2+y^2=1$

$r^2=1$

$\boxed{r=1}$



$$I = \int_0^{\pi}\int_0^1 \underbrace{e^{r^2}\,r\,dr}_{u=r^2}\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}e^{r^2}\Big|_{r=0}^{r=1} d\theta = \int_0^{\pi} \frac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1)$$

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EXAMPLE 4 Evaluate the integral

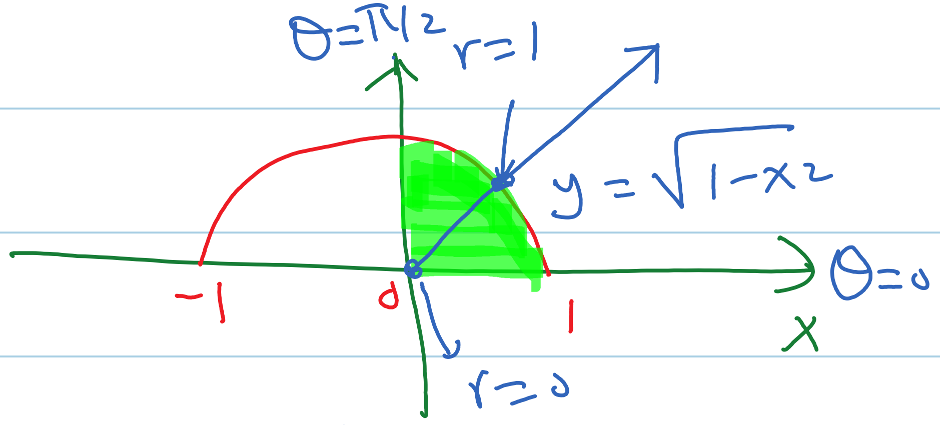
$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

$0 \le y \le \sqrt{1-x^2}$

$\boxed{0 \le x \le 1}$

$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$



$y^2 = 1 - x^2$

$y^2 + x^2 = 1$

$r^2 = 1 \Rightarrow \boxed{r = 1}$

$$\therefore I = \int_0^{\pi/2}\int_0^1 \underbrace{r^2 \cdot r\, dr\, d\theta}$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{1}{4}\theta\Big|_0^{\pi/2} = \frac{1}{4}\left(\frac{\pi}{2}\right)$$

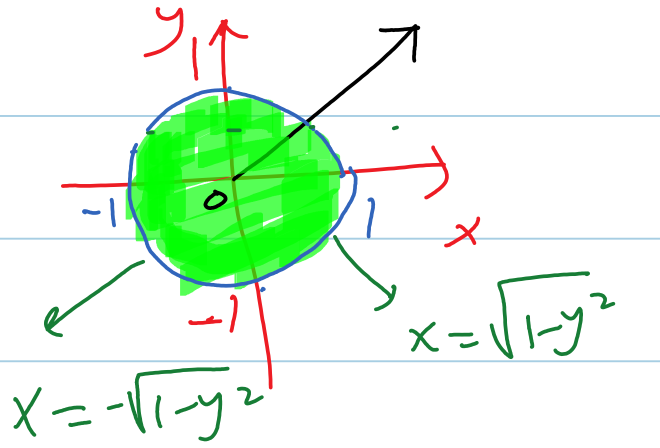
$$= \frac{\pi}{8}.$$

EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

$\boxed{x^2+y^2=1}$

$x=\pm\sqrt{1-y^2}$

$$\text{Volume} = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(9\underbrace{-x^2-y^2}\right)\boxed{dx}\,dy$$



$$= \int_0^{2\pi}\int_0^1 \underbrace{(9-r^2)\,r\,dr}\,d\theta$$

$x^2+y^2=1$

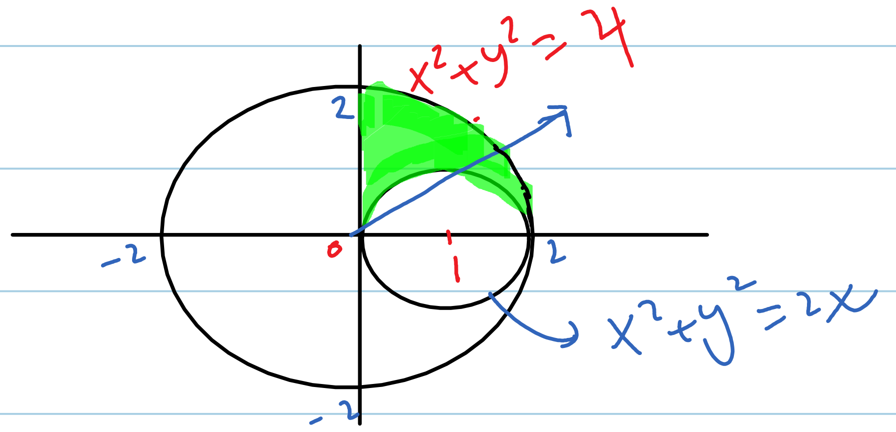
$r^2=1$

$\boxed{r=1}$

$$= \int_0^{2\pi} 9\frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} - \frac{1}{4}\right) d\theta = \frac{17}{4}\theta\Big|_0^{2\pi} = \frac{17\pi}{2}$$

Ex. Evaluate $\iint_R y\,dA$, where R lies in the first quadrant lies between $x^2+y^2=4$ and $\underbrace{x^2+y^2=2x}$.



$(x-1)^2+y^2=1$

center $(1,0)$

radius $=1$

$x^2+y^2=2x$

$r^2 = 2r\cos\theta$ Inner

$\boxed{r=0}$ or $\boxed{r=2\cos\theta}$, outer $x^2+y^2=4$

origin.

$r^2=4 \Rightarrow \boxed{r=2}$

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$$2\cos\theta \le r \le 2 \quad , \quad 0 \le \theta \le \frac{\pi}{2}$$

$$I = \iint_R y\, dA = \int_0^{\pi/2}\int_{2\cos\theta}^{2} r\sin\theta \cdot r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^3}{3}\sin\theta \Big|_{r=2\cos\theta}^{r=2} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{8}{3}\sin\theta - \frac{8}{3}\cos^3\theta\sin\theta\right) d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \sin\theta\, d\theta - \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\sin\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$\theta = 0 \Rightarrow u = 1$

$\theta = \pi/2 \Rightarrow u = 0$

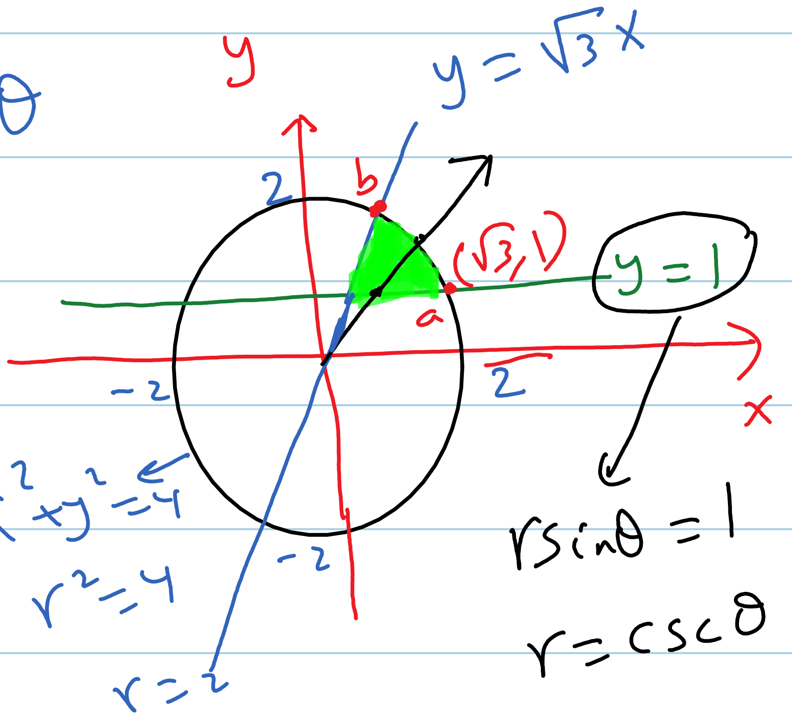
$$= \frac{-8}{3}\cos\theta \Big|_0^{\pi/2} - \frac{8}{3}\int_0^1 u^3\, du$$

$$= -\frac{8}{3}(0-1) - \frac{8}{3}\frac{u^4}{4}\Big|_0^1 = \frac{8}{3} - \frac{2}{3}(1-0)$$

$$= 2.$$

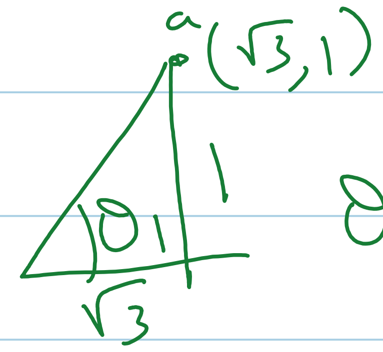
EXAMPLE 6 Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Sol. $$\text{Area} = \iint_R dA = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta$$



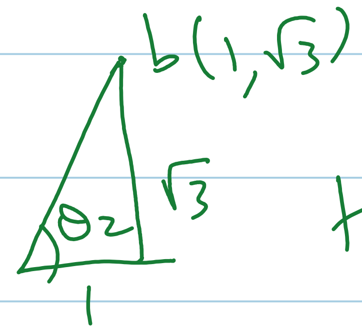
$r\sin\theta = 1$

$r = \csc\theta$



$\theta_1 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$



$\tan\theta_2 = \sqrt{3}$

$\theta_2 = \pi/3$

a: $x^2+y^2=4$, $y=1$

$\boxed{x=\sqrt{3}}$, $(\sqrt{3},1)$

b: $x^2+y^2=4$, $y=\sqrt{3}x$

$x^2+3x^2=4 \Rightarrow x^2=1$

$x=1$

$(1,\sqrt{3})$

$$\text{Area} = \int_{\pi/6}^{\pi/3} \frac{r^2}{2}\Big|_{\csc\theta}^{2} d\theta$$

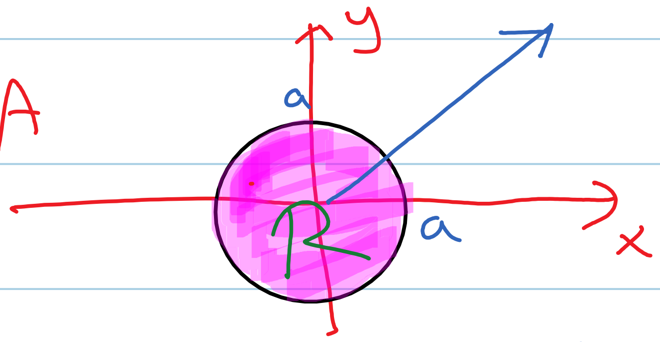
$$= \int_{\pi/6}^{\pi/3}\left(2 - \frac{\csc^2\theta}{2}\right) d\theta$$

$$= 2\theta + \frac{\cot\theta}{2}\Big|_{\pi/6}^{\pi/3}$$

$$= \left(\frac{2\pi}{3} + \frac{1}{2\sqrt{3}}\right) - \left(\frac{2\pi}{6} + \frac{\sqrt{3}}{2}\right) = \frac{\pi - \sqrt{3}}{3}$$

Ex. Find the average value of $f(x,y)=\sqrt{x^2+y^2}$ above the disk $x^2+y^2 \le a^2$ in the xy-plane.

Sol. $$av(f) = \frac{1}{Area(R)} \iint_R f(x,y)\,dA$$



$$Area(R) = \iint_R dA = \int_0^{2\pi}\int_0^a r\,dr\,d\theta$$

$x^2+y^2=a^2$
$r^2=a^2$
$\boxed{r=a}$

$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_0^a d\theta$$

$$= \int_0^{2\pi} \frac{a^2}{2} d\theta = \frac{a^2}{2}\theta \Big|_0^{2\pi}$$

$$= \pi a^2.$$

$$\iint_R f(x,y)\,dA = \iint_R \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^a \sqrt{r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\Big|_0^a d\theta = \int_0^{2\pi} \frac{a^3}{3} d\theta$$

$$= \frac{2\pi}{3}a^3.$$

$$\therefore av(f) = \frac{1}{Area}\iint_R f(x,y)\,dA = \frac{1}{\pi a^2}\cdot\frac{2\pi}{3}a^3 = \frac{2a}{3}$$

Ex. Use your knowledge in 15.4 to prove

$$\boxed{I=\int_0^{\infty} e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}}.$$

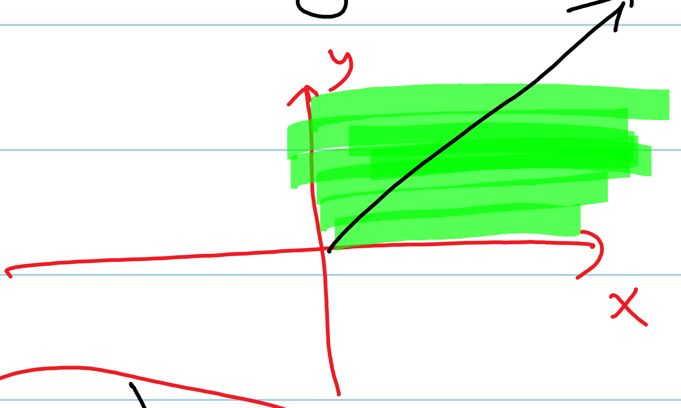
Sol.

$$I^2=\left(\int_0^{\infty} e^{-x^2}\,dx\right)\cdot\left(\int_0^{\infty} e^{-y^2}\,dy\right)$$

$$=\int_0^{\infty}\int_0^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$0 \le y < \infty$

$0 \le x < \infty$



$$I^2=\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\left(\lim_{A\to\infty}\int_0^{A} e^{-r^2}\,r\,dr\right)d\theta$$

$u=-r^2$

$du=-2r\,dr$

$-\frac{1}{2}du=r\,dr$

$$=\int_0^{\pi/2}\left(\lim_{A\to\infty} -\frac{1}{2}e^{-r^2}\Big|_0^{A}\right)d\theta$$

$\to 0$ as $A\to\infty$

$$=\int_0^{\pi/2}\left[\lim_{A\to\infty}\left(-\frac{1}{2}e^{-A^2}+\frac{1}{2}\right)\right]d\theta$$

$$=\int_0^{\pi/2}\left(0+\frac{1}{2}\right)d\theta=\frac{1}{2}\theta\Big|_0^{\pi/2}=\frac{\pi}{4}$$

$$\therefore\ I^2=\pi/4$$

$$\Rightarrow I=\int_0^{\infty} e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}.$$

∎

Review (12.6)

الرسومات.

15.5 Triple Integrals in Rectangular Coordinates

Triple Integrals

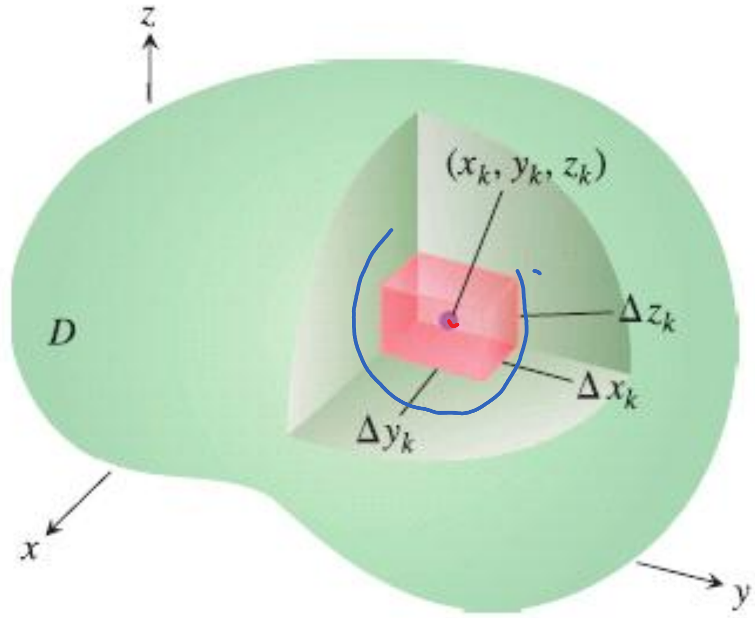


FIGURE 15.29 Partitioning a solid with rectangular cells of volume ΔV_k.

$$\boxed{S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta V_k.}$$

We are interested in what happens as D is partitioned by smaller and smaller cells, so that Δx_k, Δy_k, Δz_k and the norm of the partition $\|P\|$, the largest value among Δx_k, Δy_k, Δz_k, all approach zero. When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is **integrable** over D. As before, it can be

shown that when F is continuous and the bounding surface of D is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then F is integrable. As $\|P\| \to 0$ and the number of cells n goes to ∞, the sums S_n approach a limit. We call this limit the **triple integral of F over D** and write

$$\lim_{n\to\infty} S_n = \iiint_D F(x, y, z)\, dV \quad \text{or} \quad \lim_{\|P\|\to 0} S_n = \iiint_D F(x, y, z)\, dx\, dy\, dz.$$

Volume of a Region in Space

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

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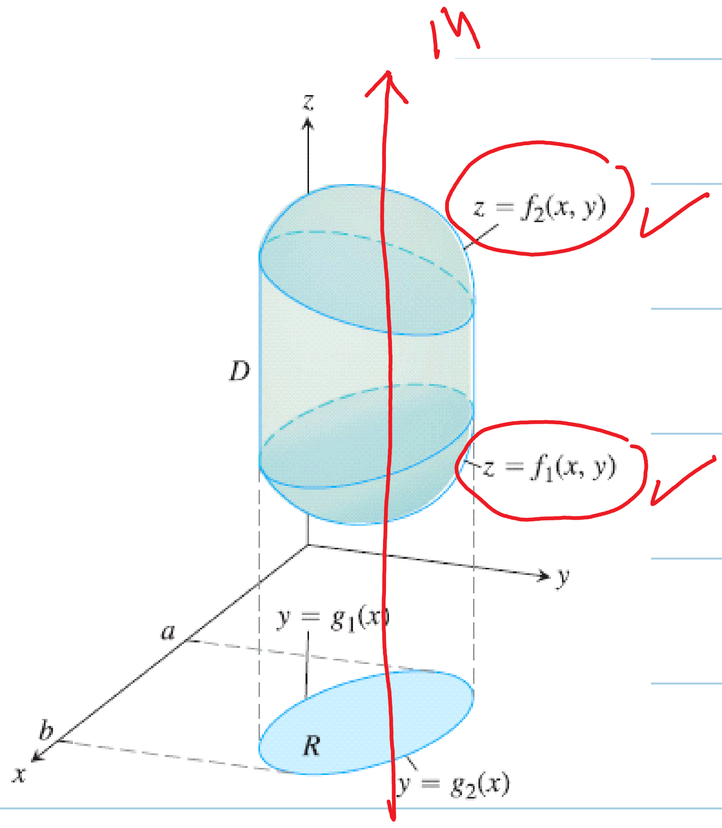
Finding Limits of Integration in the Order *dz dy dx*

To evaluate

$$\iiint_D F(x, y, z)\, dV$$

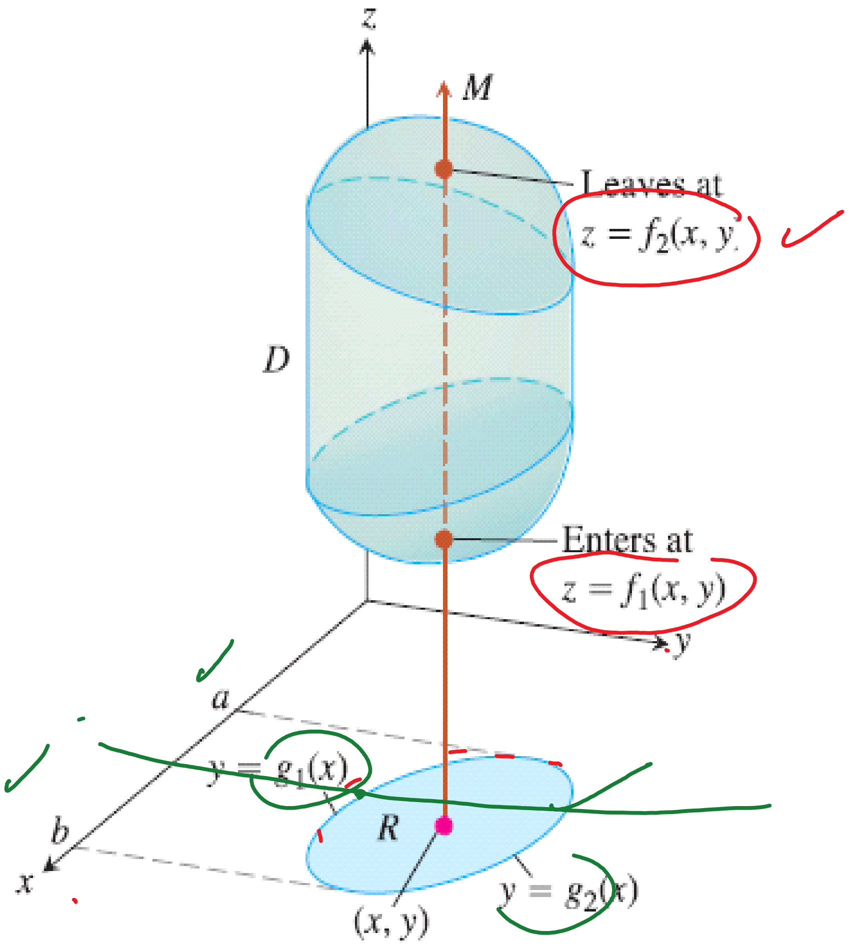
dzdydx

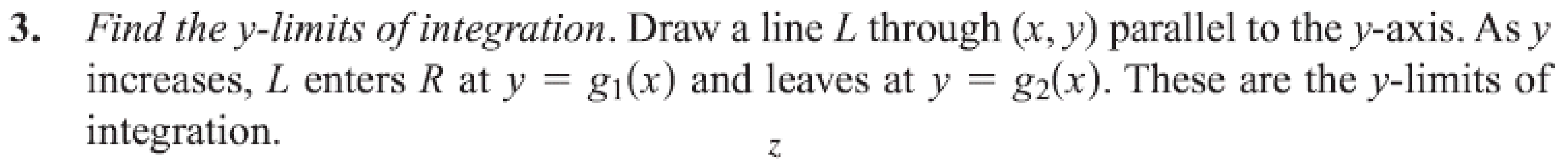
1. *Sketch*. Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



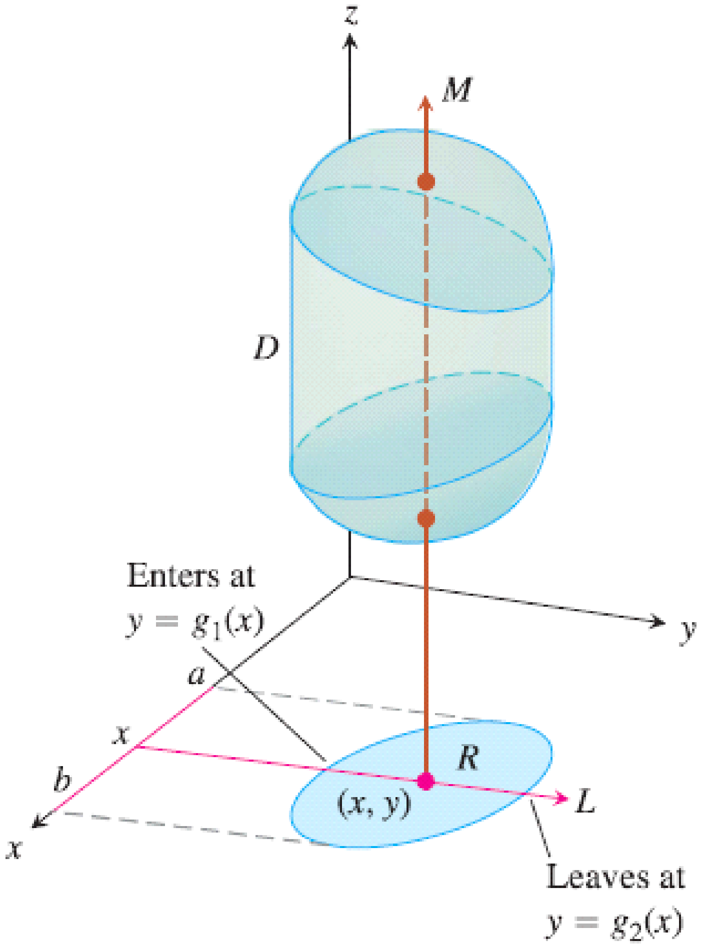
2. *Find the z-limits of integration*. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.

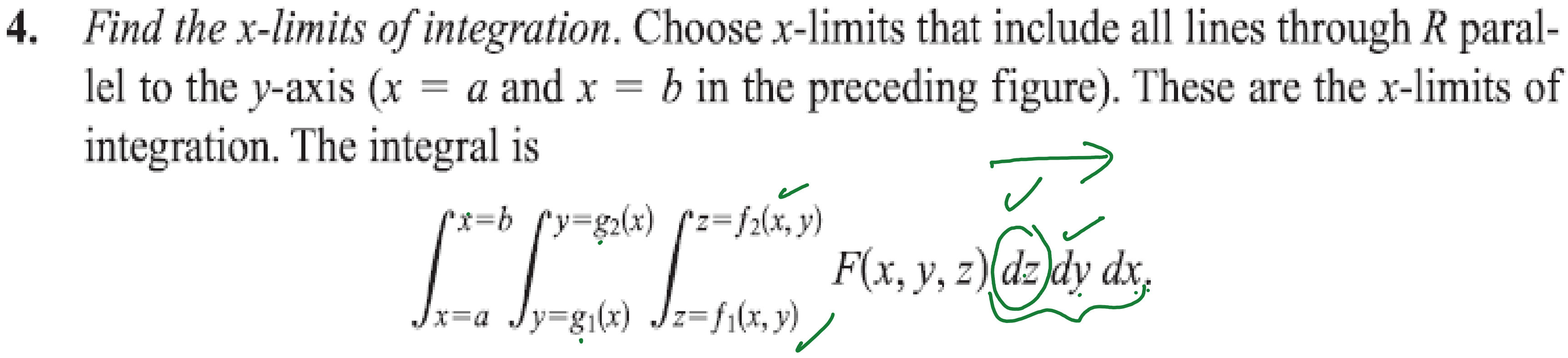
dzdydx





3. *Find the y-limits of integration*. Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.





4. *Find the x-limits of integration*. Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\,dz\,dy\,dx.$$

Ex ①. Evaluate $I=\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx$

Sol. $I=\int_0^1\int_0^{1-x^2} xz\Big|_{z=3}^{z=4-x^2-y}\,dy\,dx$

$=\int_0^1\int_0^{1-x^2}\left(x(4-x^2-y)-3x\right)dy\,dx$ (Double integral)

$$= \int_0^1 \int_0^{1-x^2} (x - x^3 - xy)\, dy\, dx$$

$$= \int_0^1 xy - x^3 y - \frac{xy^2}{2} \Big|_{y=0}^{y=1-x^2} dx$$

$$= \int_0^1 \left(x(1-x^2) - x^3(1-x^2) - \frac{x}{2}(1-x^2)^2 \right) dx$$

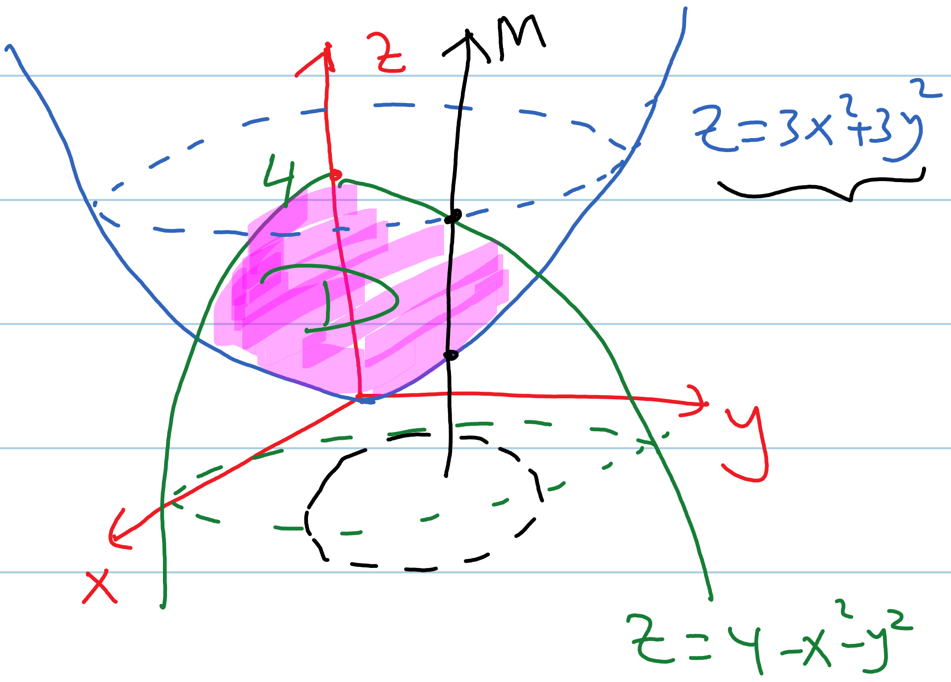
$= \cdots\cdots$

Ex2. Find the volume of the solid bounded by

$z = 3x^2+3y^2$ and $z = 4-x^2-y^2$

Sol.

$z = x^2+y^2$



$$\text{Volume} = \iiint_D dV$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{3x^2+3y^2}^{4-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x2}}^{\sqrt{1-x2}} z \Big|_{3x^2+3y^2}^{4-x^2-y^2} dy\, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x2}}^{\sqrt{1-x^2}} \left[4-4(x^2+y^2)\right] dy\, dx$$

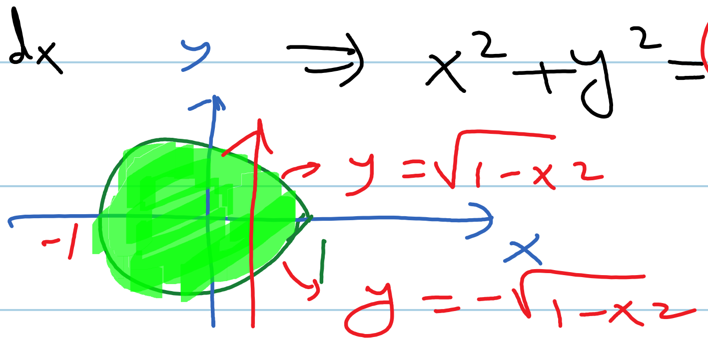
$$= \int_0^{2\pi} \int_0^1 (4-4r^2)\, r\, dr\, d\theta \, .$$

The surfaces intersect on the circular cylinder:

$z = 3x^2+3y^2$, $z = 4-x^2-y^2$

$4-x^2-y^2 = 3x^2+3y^2$

$\Rightarrow x^2+y^2 = \boxed{1}$



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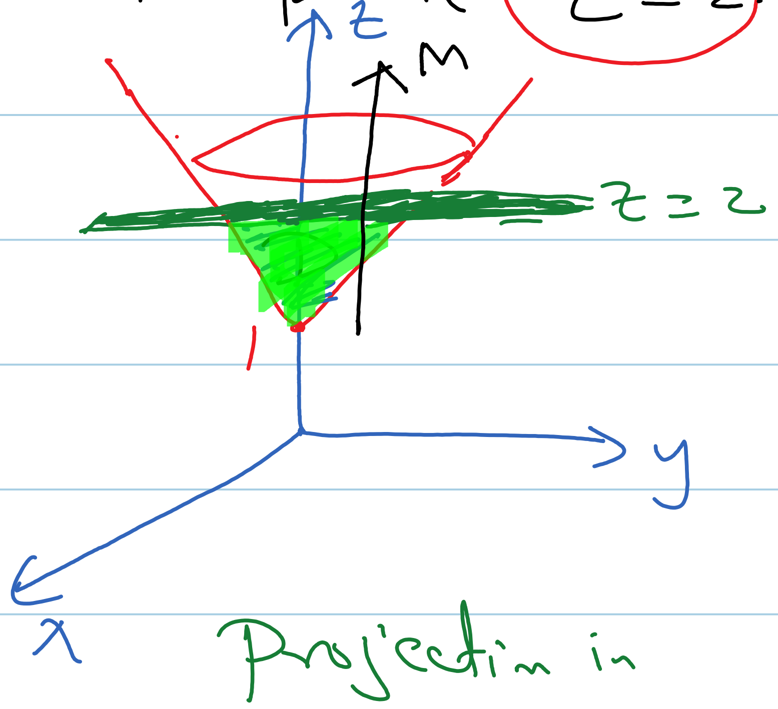
Ex 3. Find the volume of the solid bounded by $-x^2-y^2+z^2=1$ and the plane $\boxed{z=2}$.



Sol. $z^2 = 1+x^2+y^2$

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \int_{\sqrt{1+x^2+y^2}}^{2} dz\,dx\,dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \left(2-\sqrt{1+x^2+y^2}\right) dx\,dy$$

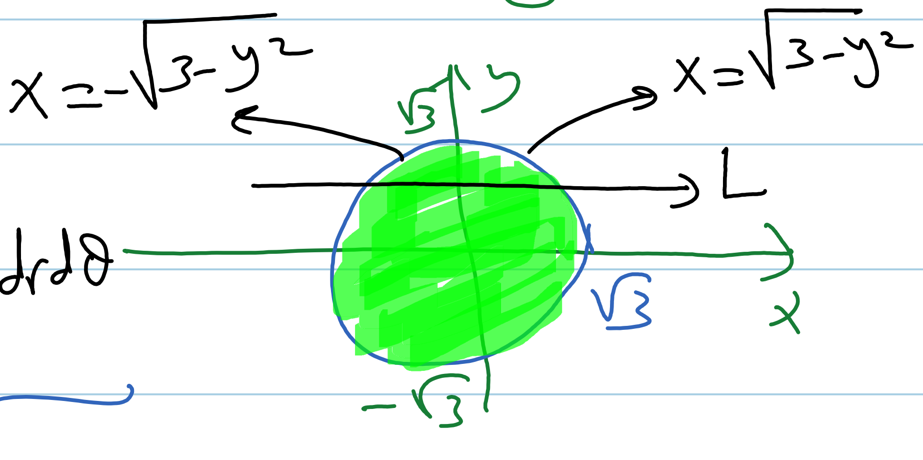
Projection in xy-plane

$z=2,\ x^2+y^2+1=z^2$

$\Rightarrow x^2+y^2+1=4$

$x^2+y^2=3$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \left(2-\sqrt{1+r^2}\right) r\,dr\,d\theta$$



$$= \underbrace{\int_0^{2\pi} \int_0^{\sqrt{3}} 2r\,dr\,d\theta}_{\int_0^{2\pi} r^2\Big|_0^{\sqrt{3}} d\theta} - \underbrace{\int_0^{2\pi} \int_0^{\sqrt{3}} r\sqrt{1+r^2}\,dr\,d\theta}_{u=1+r^2}$$

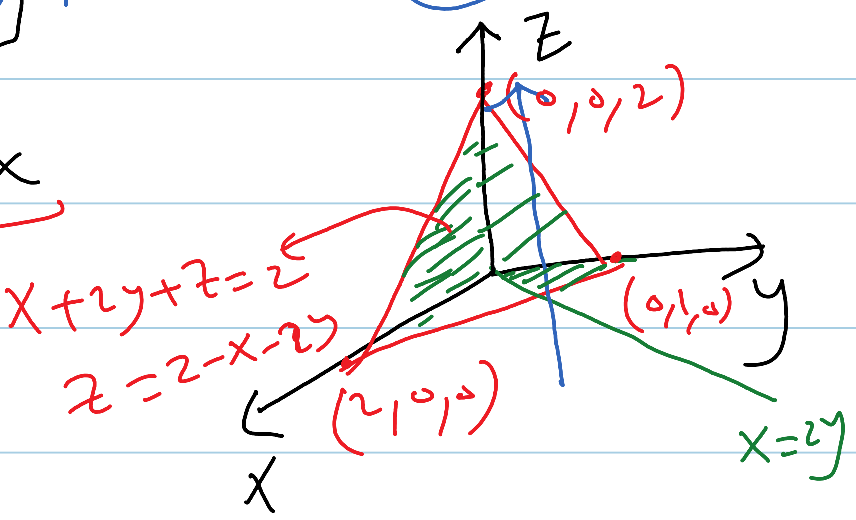
$\vdots$

$= 6\pi.$

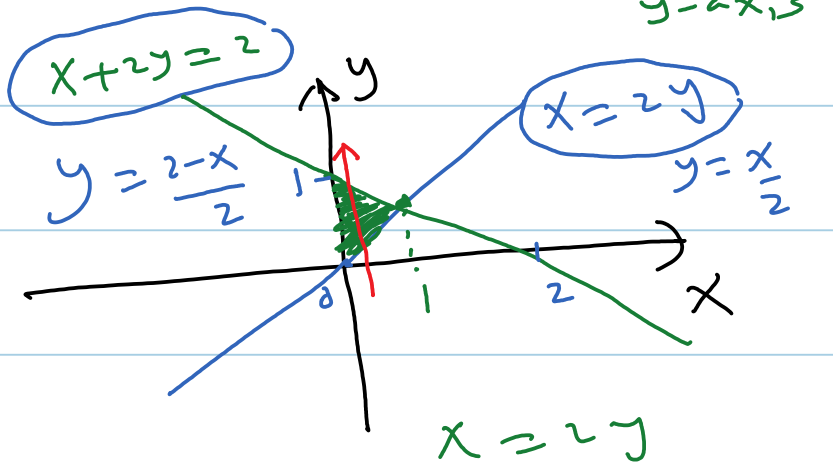
Ex4. Find the volume of the tetrahedron bounded by $\boxed{x+2y+z=2}$, $\boxed{x=2y}$, $x=0$, $\boxed{z=0}$

Sol.

$$V=\int_0^1\int_{\frac{x}{2}}^{1-\frac{x}{2}}\int_0^{2-x-2y} dz\,dy\,dx$$



In xy-plane $x+2y=2$, $x=2y$, $\underbrace{x=0}_{y\text{-axis}}$



$x=2y$

$x=2-2y$

$2y=2-2y$

$4y=2$

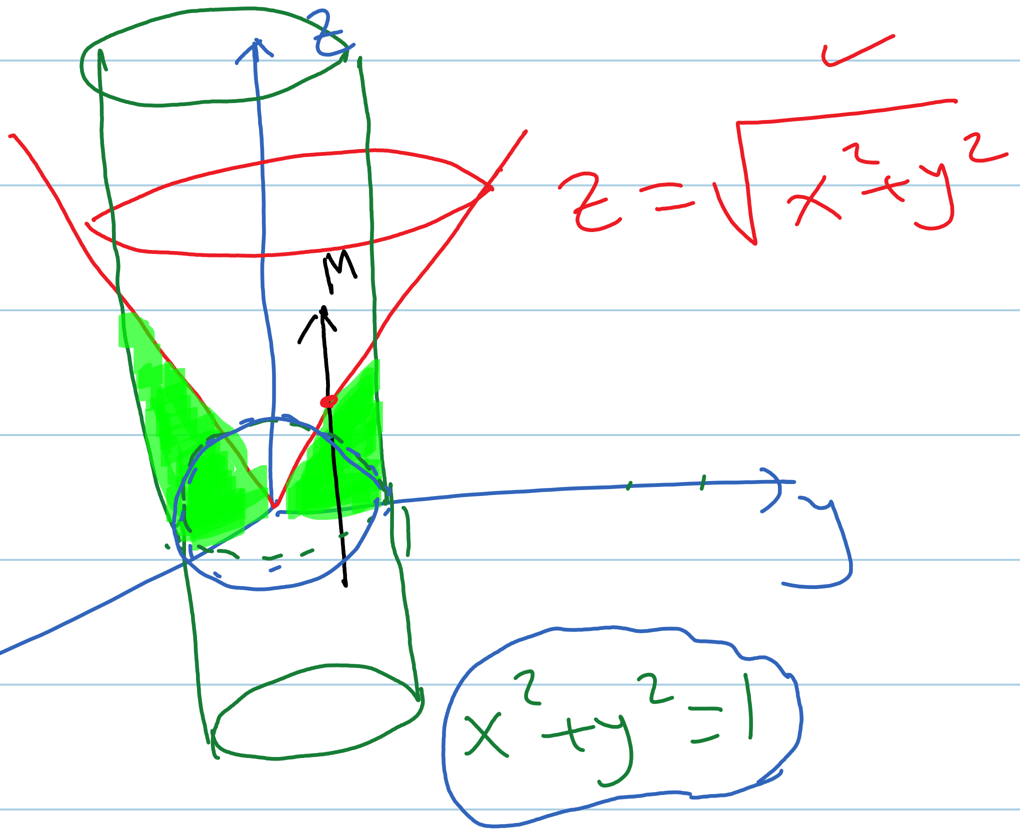
$\boxed{y=\frac{1}{2}}$

$\boxed{x=1}$

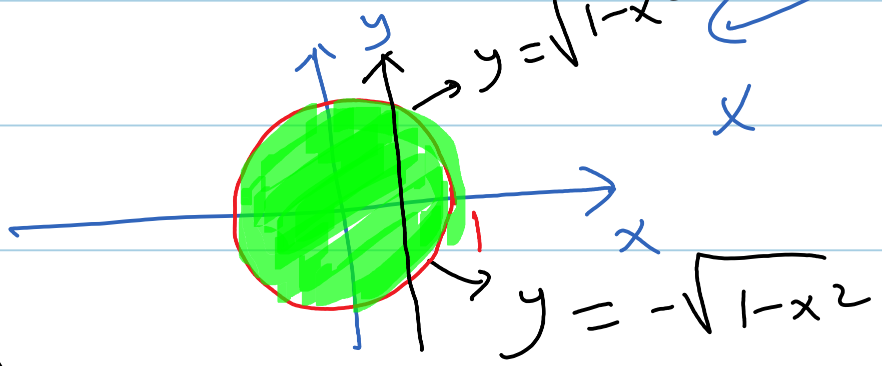
Ex5. Find the volume of the region in space bounded below by the xy-plane, Laterally by the cylinder $x^2+y^2=1$ and above by the cone $z=\sqrt{x^2+y^2}$.

Sol.

$$V=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{x^2+y^2}} dz\,\boxed{dy\,dx}$$



In xy-plane



$$V=\int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{r} dz\,\underbrace{r\,dr\,d\theta}$$

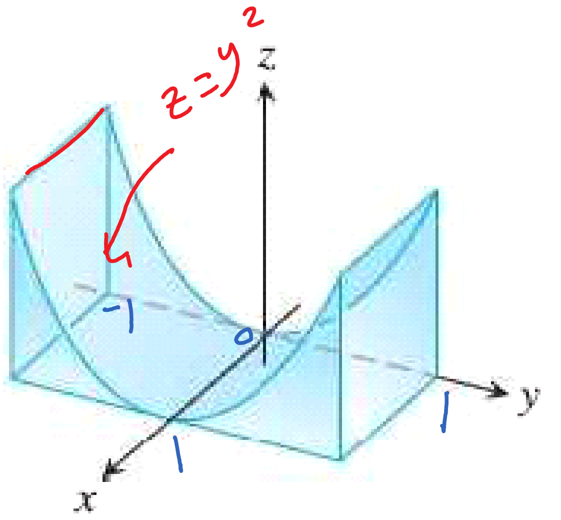
$$=\int_{0}^{2\pi}\int_{0}^{1} rz\Big|_{z=0}^{z=r} dr\,d\theta$$

$$=\int_{0}^{2\pi}\int_{0}^{1} r^2\,dr\,d\theta=\int_{0}^{2\pi}\frac{r^3}{3}\Big|_{0}^{1} d\theta$$

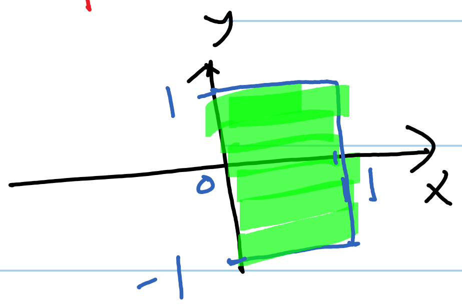
$$=\int_{0}^{2\pi}\frac{1}{3}d\theta=\frac{2\pi}{3}$$

Find the volumes of the regions in Exercises 23–36.

23. The region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0, x = 1, y = -1, y = 1$



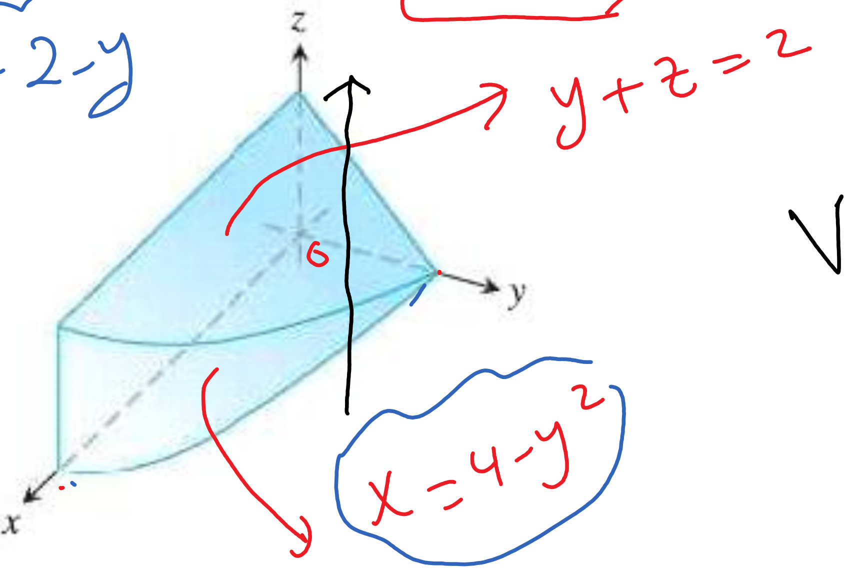
$$V = \int_0^1 \int_{-1}^{1} \int_0^{y^2} dz\,dy\,dx$$



$x, y, z \geq 0$

25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$

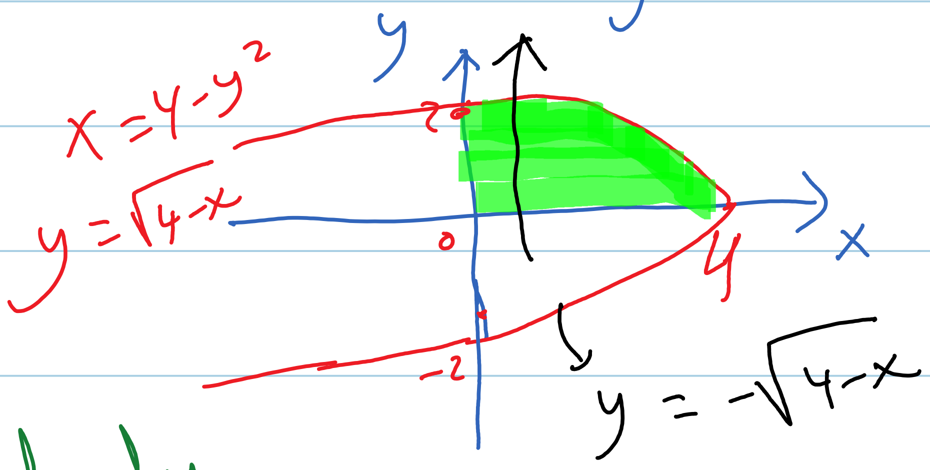
$z = 2-y$



$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\,dy\,dx$$

x-y-plane $\boxed{x=4-y^2}$

$y = 2$



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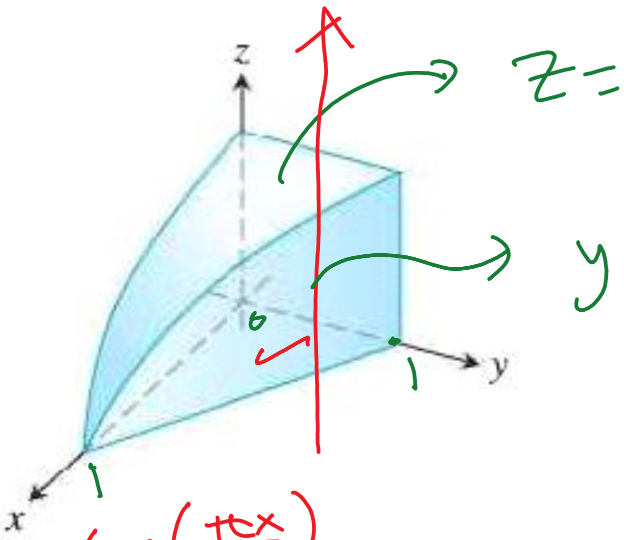
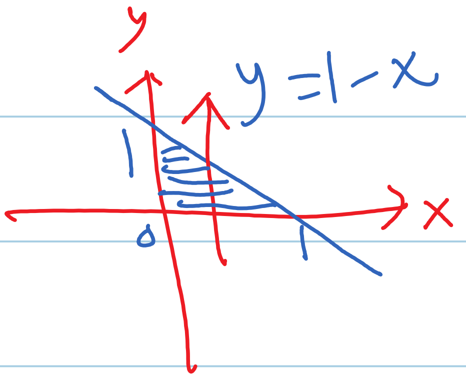
$$V = \int_0^2 \int_0^{4-y^2} \int_0^{2-y} dz\,dx\,dy$$

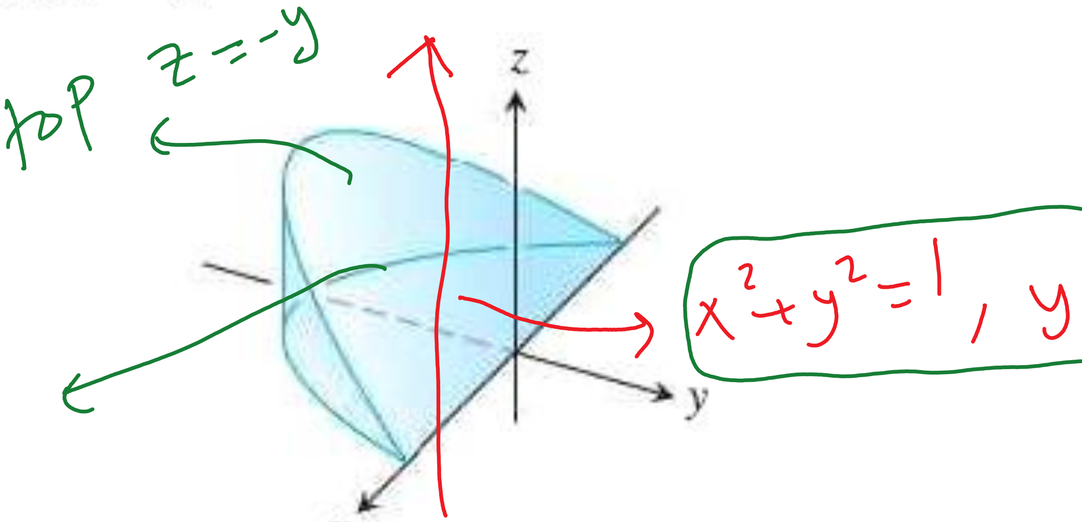
28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2)$, $0 \le x \le 1$



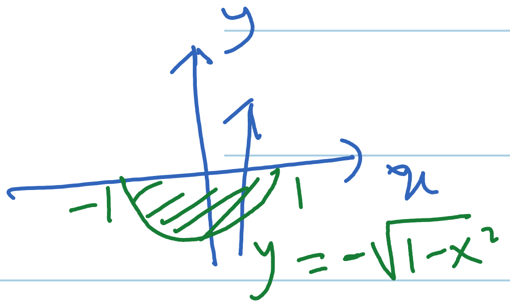
$$\text{Volume} = \int_0^1 \int_0^{1-x} \int_0^{\cos(\frac{\pi x}{2})} dz\,dy\,dx$$



26. The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$

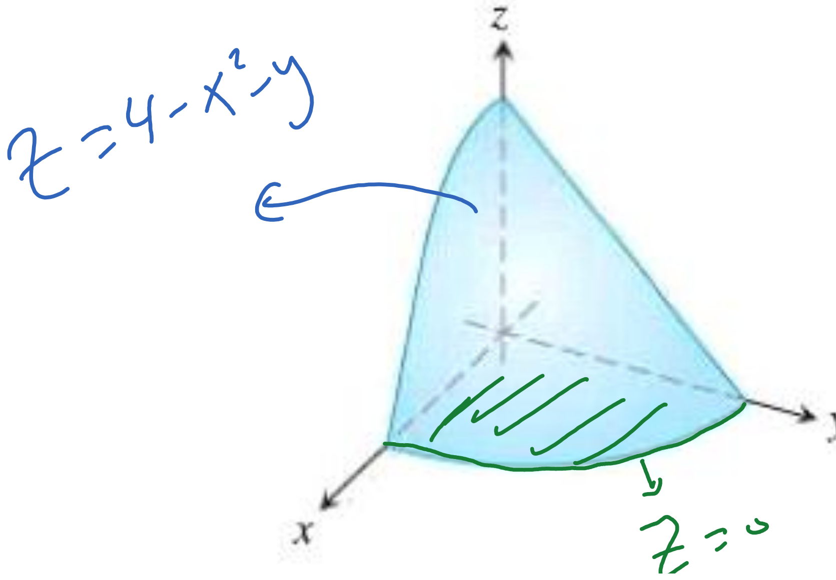


$$\text{Volume} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_0^{-y} dz\,dy\,dx$$

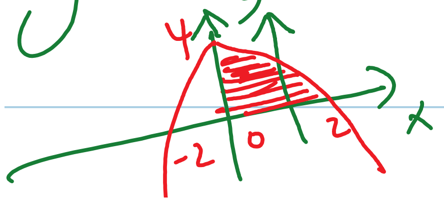


30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$

$$V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\,dy\,dx$$

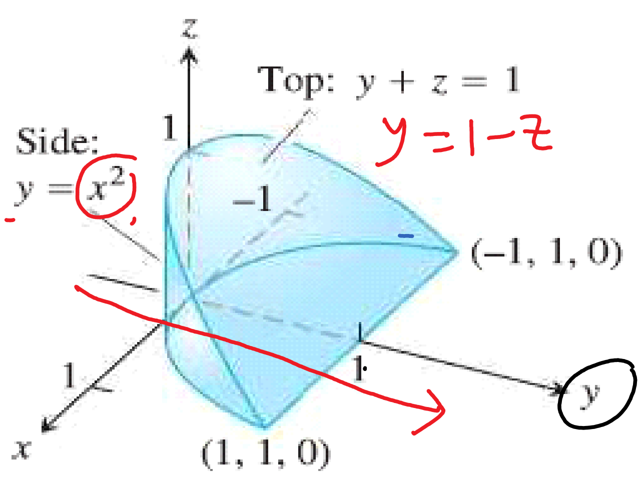


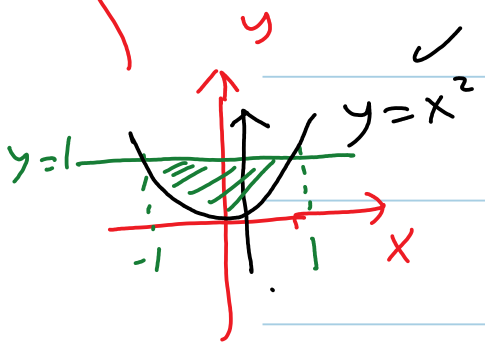
$4 - x^2 - y = 0 \Rightarrow y = 4 - x^2$



21. Here is the region of integration of the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx.$$





Rewrite the integral as an equivalent iterated integral in the order

a. $dy\,dz\,dx$ **b.** $dy\,dx\,dz$

c. $dx\,dy\,dz$ **d.** $dx\,dz\,dy$

[✓ **e.** $dz\,dx\,dy$]

ⓔ $\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dz\,dx\,dy$

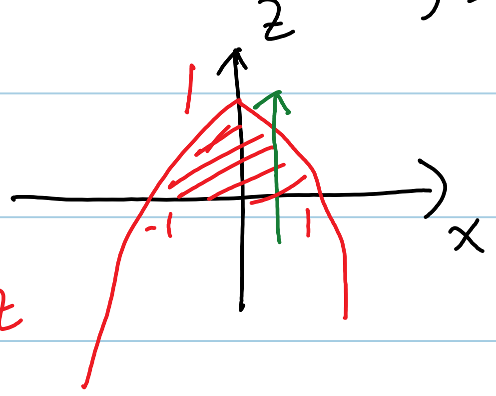
ⓐ $V = \int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} \boxed{dy}\,\underbrace{dz\,dx}$ ✓

xz-plane

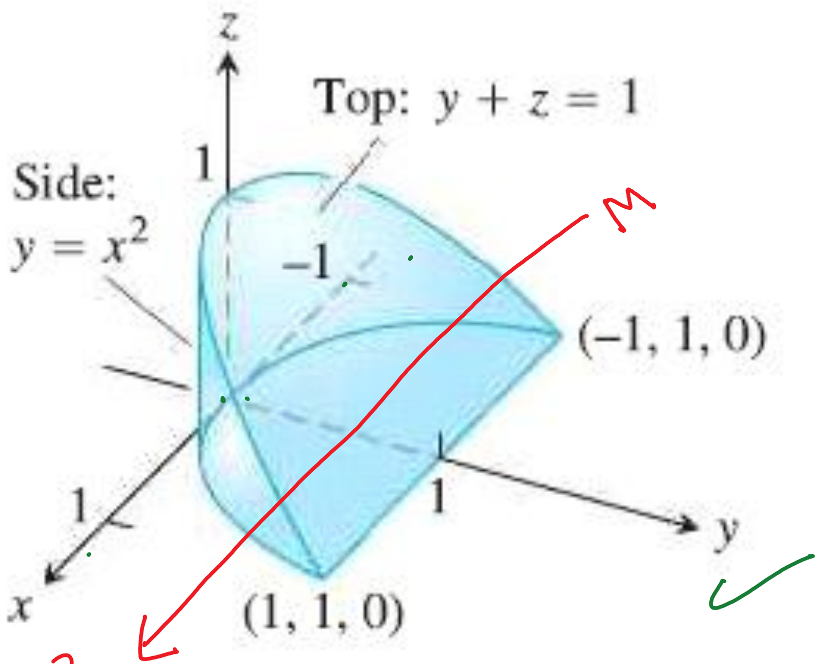
$y = x^2, \quad y + z = 1$

$x^2 + z = 1$

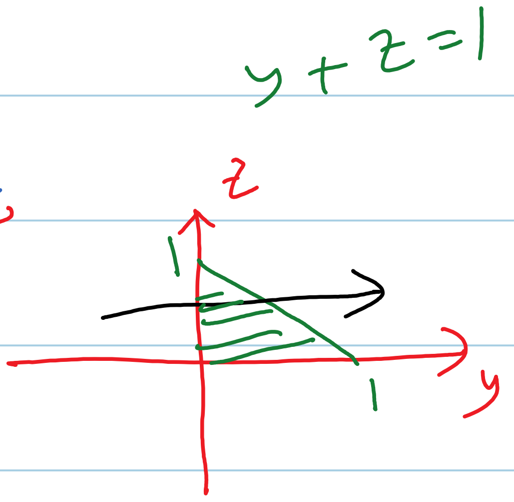
$z = 1 - x^2$



ⓑ $V = \int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$



ⓒ $$V = \int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, \underbrace{dy\, dz}$$



ⓓ $$V = \int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dz\, dy$$

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\textbf{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\, dV.$$

EXAMPLE 4 Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

$\int_0^2\int_0^2\int_0^2 dz\,dy\,dx$ ✓ $0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 2$

Solution

The volume of the region D is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\int_0^2\int_0^2\int_0^2 \overset{F}{\boxed{xyz}}\, dx\, dy\, dz = \int_0^2\int_0^2 \left[\frac{x^2}{2} yz\right]_{x=0}^{x=2} dy\, dz = \int_0^2\int_0^2 2yz\, dy\, dz$$

$$= \int_0^2 \left[y^2 z\right]_{y=0}^{y=2} dz = \int_0^2 4z\, dz = \left[2z^2\right]_0^2 = 8.$$

$$\begin{matrix}\text{Average value of}\\ xyz \text{ over the cube}\end{matrix} = \frac{1}{\text{volume}} \iiint_{\text{cube}} xyz\, dV = \left(\frac{1}{8}\right)(8) = 1.$$

15.6 ∝

15.7 Triple Integrals in Cylindrical and Spherical Coordinates

Integration in Cylindrical Coordinates ✓

(x, y, z) $x = r\cos\theta$
$y = r\sin\theta$
$z = z$ ✓

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

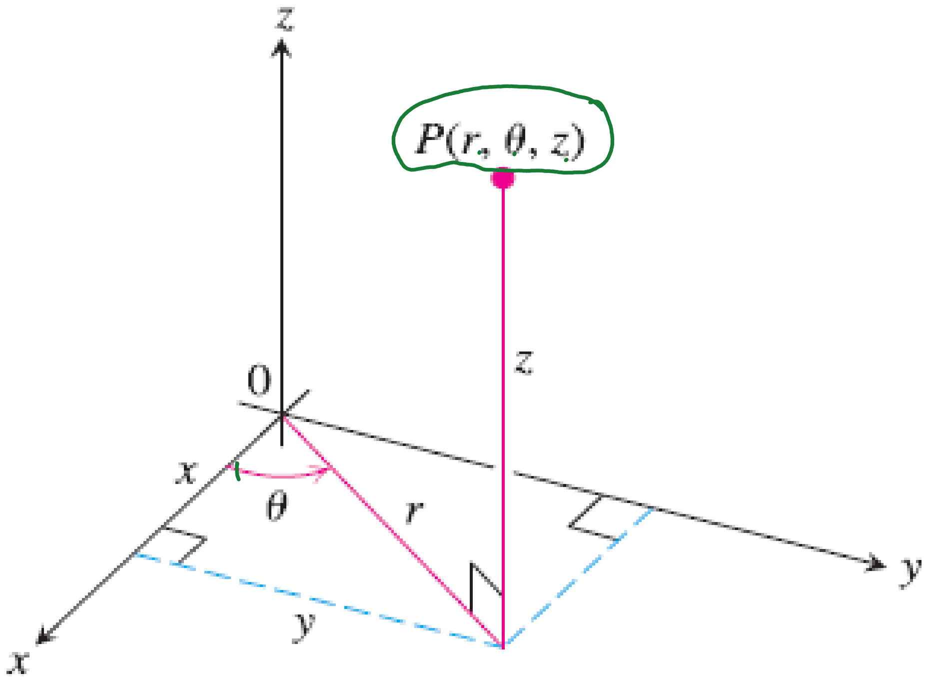
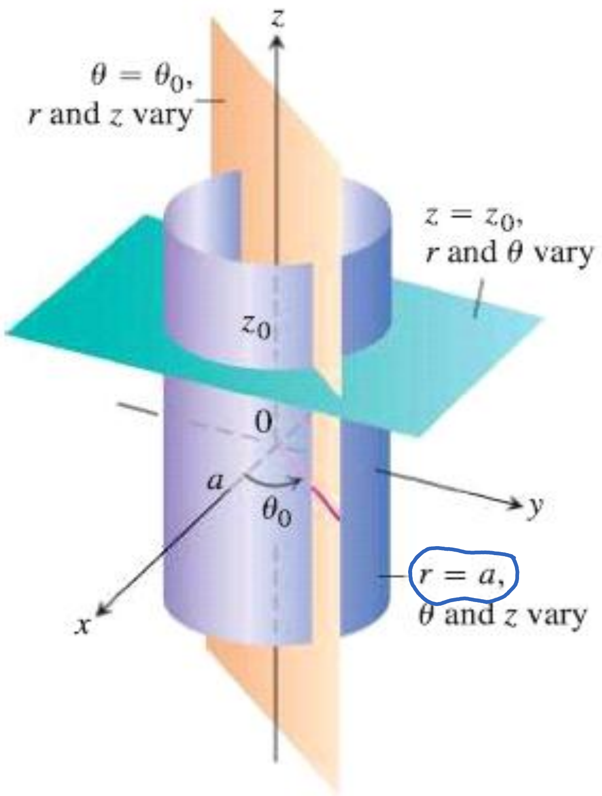


FIGURE 15.42 The cylindrical coordinates of a point in space are r, θ, and z.

(x, y)

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$



$r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis

$\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis

$z = z_0$ describes a plane perpendicular to the z-axis.

Example. $\boxed{r = 4}$ Cylinder, radius 4, axis the z-axis

$\boxed{\theta = \frac{\pi}{3}}$ Plane containing the z-axis

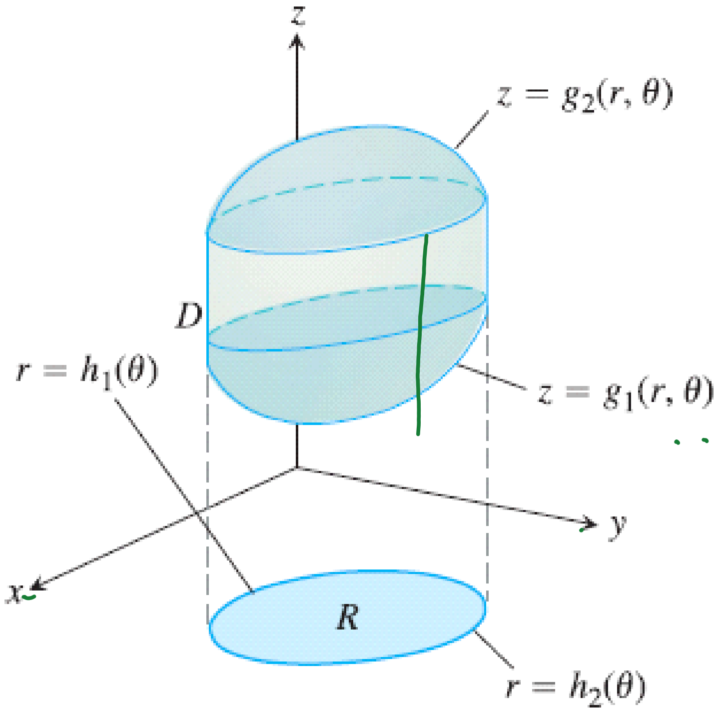
$\boxed{z = 2.}$ Plane perpendicular to the z-axis

How to Integrate in Cylindrical Coordinates

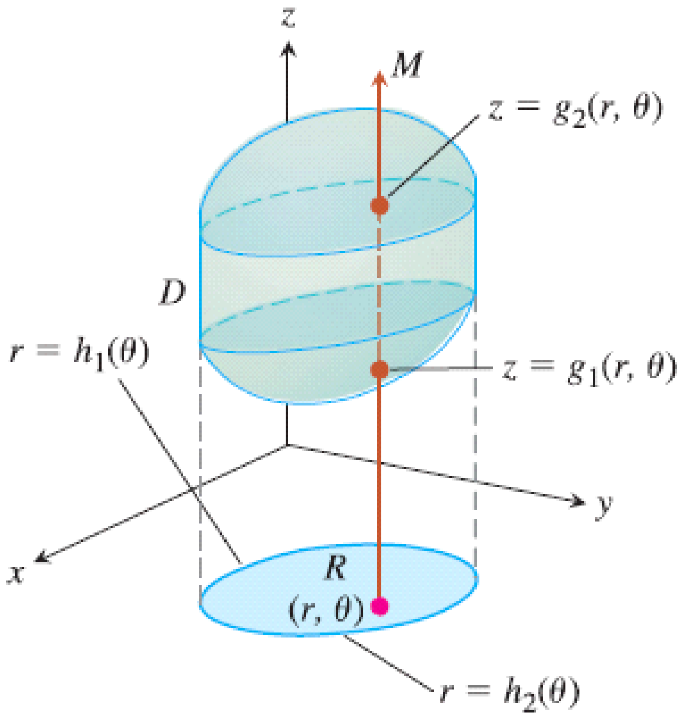
To evaluate

$$\iiint_D f(r, \theta, z)\, dV$$

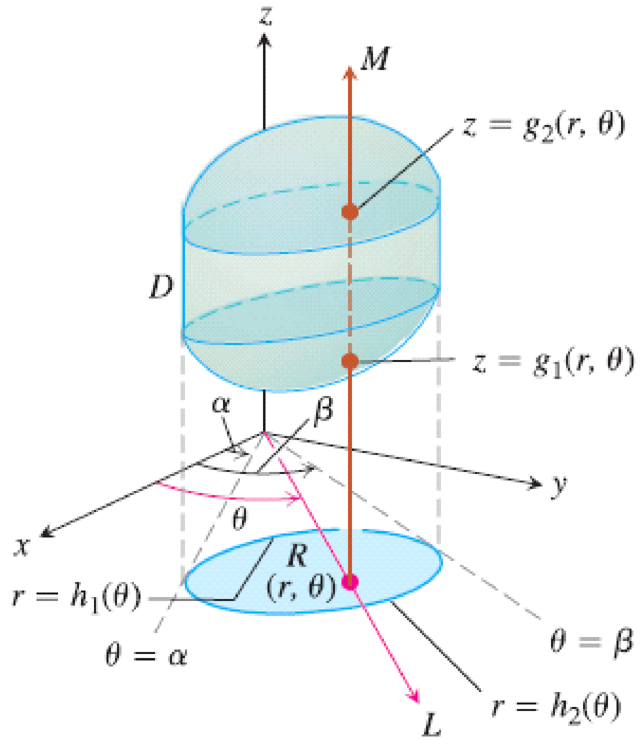
1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



2. *Find the z-limits of integration*. Draw a line M through a typical point (r, θ) of parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves a $z = g_2(r, \theta)$. These are the z-limits of integration.



3. *Find the r-limits of integration.* Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.

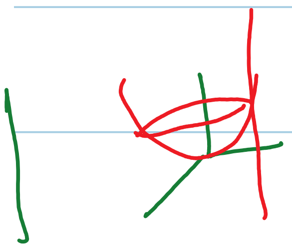


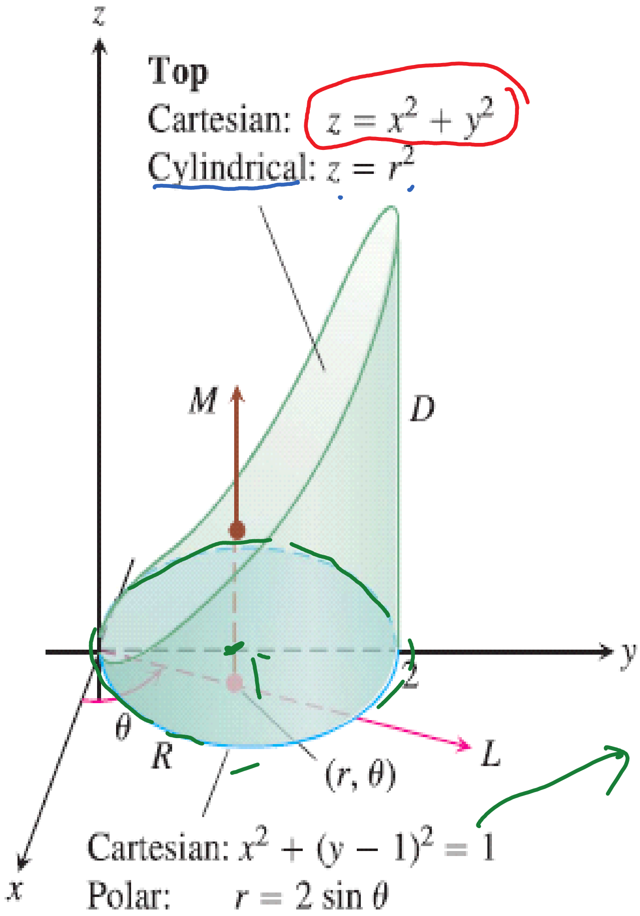
4. *Find the θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} \boxed{f(r, \theta, z)}\, dz\, r\, dr\, d\theta.$$

EXAMPLE 1 Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

$(0,1)$ radius $=1$





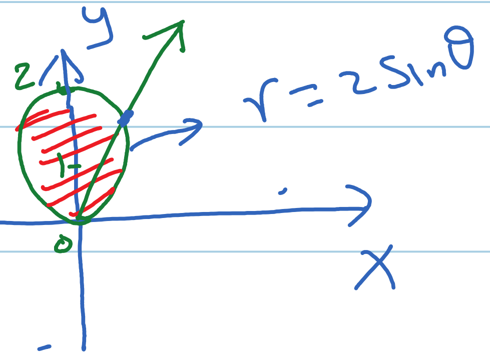
$$\iiint_D f(x,y,z)\, dV$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} g(r,\theta,z)\, dz\, r\, dr\, d\theta$$

$x^2+y^2-2y+1=1$

$r^2-2r\sin\theta=0$

$r=0,\ r=2\sin\theta$



$r=2\sin\theta$

Ex. Convert into Cylindrical:

$$I=\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}\left(x^2+y^2\right)\underbrace{dz\,dx\,dy}_{}\ .$$

Sol.

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{0}^{r\cos\theta} r^2\,dz\,r\,dr\,d\theta$$

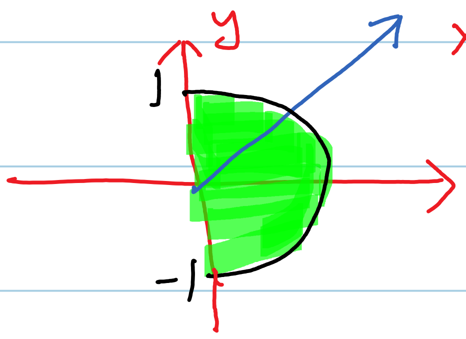
$\boxed{x=0}$ y-axis

$\rightarrow x=\sqrt{1-y^2} \Rightarrow x^2+y^2=1$

$0\le x\le\sqrt{1-y^2}$

$-1\le y\le 1$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r^3 z\Big|_{z=0}^{z=r\cos\theta} dr\,d\theta$$



$x^2+y^2=1$

$r^2=1$

$\boxed{r=1}$

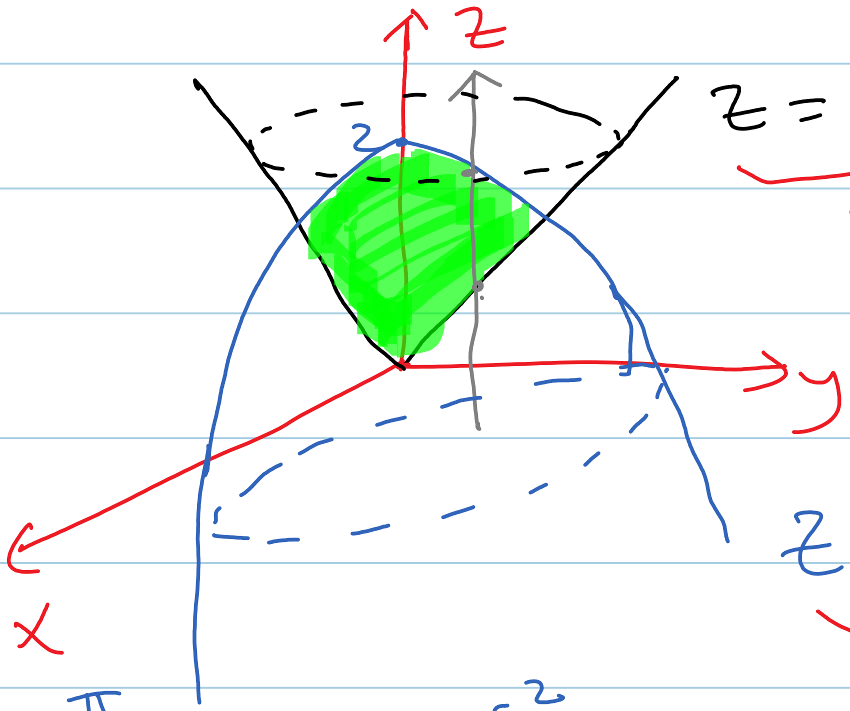
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r^4\cos\theta\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^5}{5}\cos\theta\Big|_{r=0}^{r=1} d\theta$$

$$=\frac{1}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta=\frac{1}{5}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{2}{5}.$$

Ex. Let D be the region bounded below by $z=\sqrt{x^2+y^2}$ and above by $z=2-x^2-y^2$. Find the volume of D.



$$z=\underbrace{\sqrt{x^2+y^2}}_{z=r}$$

$$\underbrace{z=2-x^2-y^2}_{z=2-r^2}$$

$$\text{Volume}=\int_0^{2\pi}\int_0^1\int_r^{2-r^2} 1\,dz\,\underbrace{r\,dr\,d\theta}$$

Projection

$z=2-r^2$

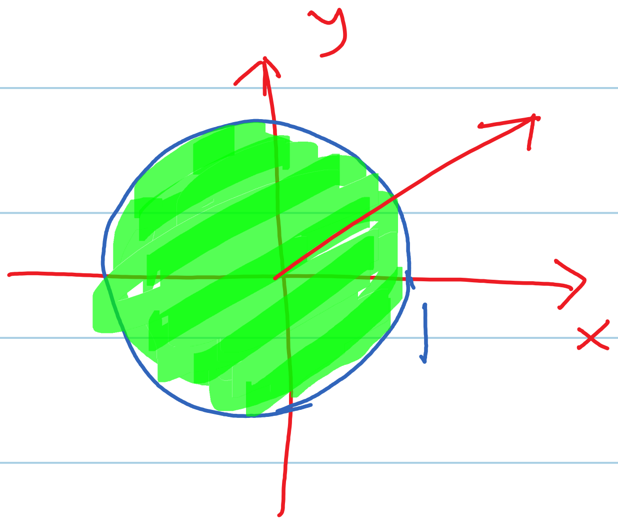
$z=r$

$2-r^2=r$

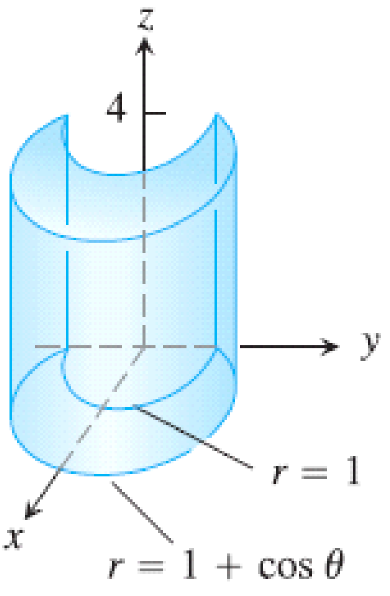
$r^2+r-2=0$

$(r+2)(r-1)=0$

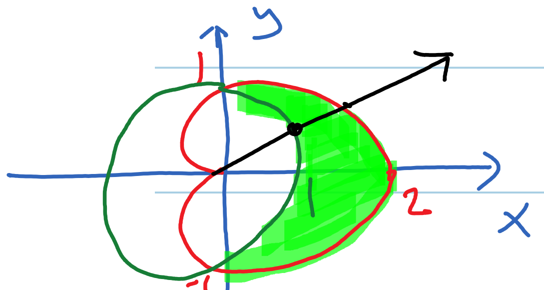
$\underbrace{r=-2}_{\text{reject}},\ r=1$



17. *D* is the solid right cylinder whose base is the region in the *xy*-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



base



$$V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta}\int_0^4 dz\,r\,dr\,d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} \int_{0}^{4} r\, dz\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} rz \Big|_{z=0}^{z=4} dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} 4r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r^2 \Big|_{r=1}^{r=1+\cos\theta} d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[(1+\cos\theta)^2 - 1\right] d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(2\cos\theta + \cos^2\theta\right) d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[2\cos\theta + \frac{1+\cos 2\theta}{2}\right] d\theta$$

$$= \left[4\sin\theta + \theta + \frac{\sin 2\theta}{2}\right]\Big|_{-\frac{\pi}{2}}^{\pi/2}$$

$$= \left(4 + \frac{\pi}{2} + 0\right) - \left(-4 - \frac{\pi}{2} - 0\right)$$

$$= 8 + \pi.$$

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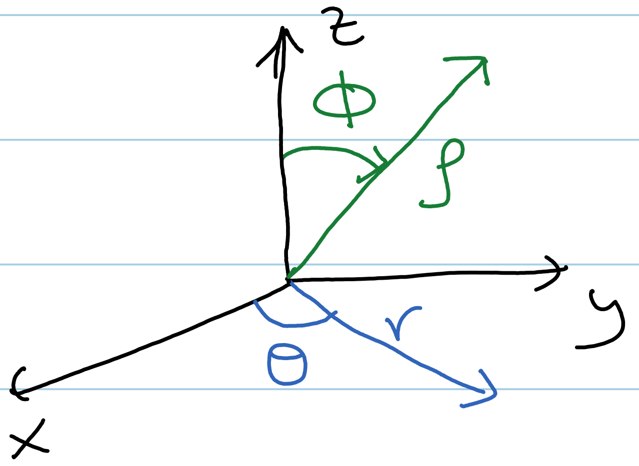
$(x,y,z) \rightsquigarrow (\rho, \phi, \theta)$

Spherical Coordinates and Integration



$$\boxed{z = \rho \cos\phi}$$

$$r = \rho \sin\phi$$

$$x = r\cos\theta$$

$$\boxed{x = \rho\sin\phi\cos\theta}$$

$$y = r\sin\theta$$

$$\boxed{y = \rho\sin\phi\sin\theta}$$

$$x^2+y^2 = r^2 = \rho^2\sin^2\phi$$

$$\boxed{x^2+y^2 = \rho^2\sin^2\phi}$$

$$x^2+y^2+z^2 = \rho^2\sin^2\phi + \rho^2\cos^2\phi$$

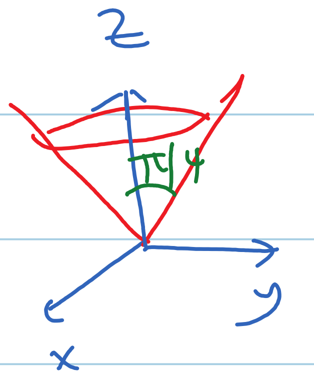
$$= \rho^2(1)$$

$$\therefore \boxed{x^2+y^2+z^2 = \rho^2}$$

$\rho > 0$, $0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$.

Ex. Convert to spherical



$$z=\sqrt{x^2+y^2}$$

Sol. $\rho\cos\phi=\sqrt{\rho^2\sin^2\phi}$

$$\rho\cos\phi=|\overset{>0}{\rho}\,\overset{>0}{\sin\phi}| \quad , \rho>0$$

$$0\le\phi\le\pi$$

$$=\rho\sin\phi$$

$$\cancel{\rho}\cos\phi=\cancel{\rho}\sin\phi \quad , \rho>0$$

$$\tan\phi=1\Rightarrow\boxed{\phi=\frac{\pi}{4}}$$

Cone

Ex. $\underbrace{x^2+y^2+z^2}=9$ sphere

$\rho^2=9\Rightarrow\boxed{\rho=3}$ sphere.

Ex. Convert to spherical coordinates.

$$x^2+y^2+(z-1)^2=1$$

Sol. $\underbrace{x^2+y^2+z^2}-2z=0$

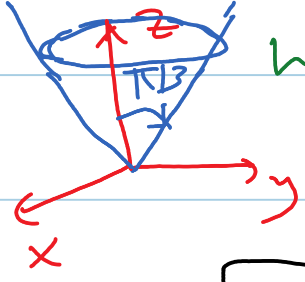
$$\rho^2-2\rho\cos\phi=0$$

$$\rho=2\cos\phi \quad , \rho>0.$$

Ex. Describe

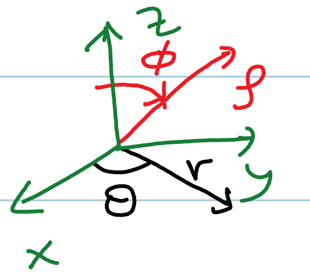
1) $\rho=4$ $\boxed{\text{Sphere}}$, radius 4, center at origin.

2) $\phi=\frac{\pi}{3}$ $\boxed{\text{Cone}}$ opening up from the origin, making an angle of $\pi/3$ radians with the positive z-axis.



3) $\theta=\frac{\pi}{3}$ $\boxed{\text{Half-plane}}$, along the z-axis, making an angle of $\frac{\pi}{3}$ with the positive x-axis.

Recall, $(x, y, z) \to (\rho, \phi, \theta)$



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin \phi, \qquad \boxed{x = r\cos\theta = \rho \sin\phi \cos\theta,}$$

$$\boxed{z = \rho \cos \phi,} \qquad y = r \sin\theta = \rho \sin\phi \sin\theta,$$

$$\rho = \sqrt{\underbrace{x^2 + y^2}_{} + z^2} = \sqrt{r^2 + z^2}.$$

$x^2 + y^2 + z^2 = \rho^2$

$\rho > 0$

$0 \le \phi \le \pi$

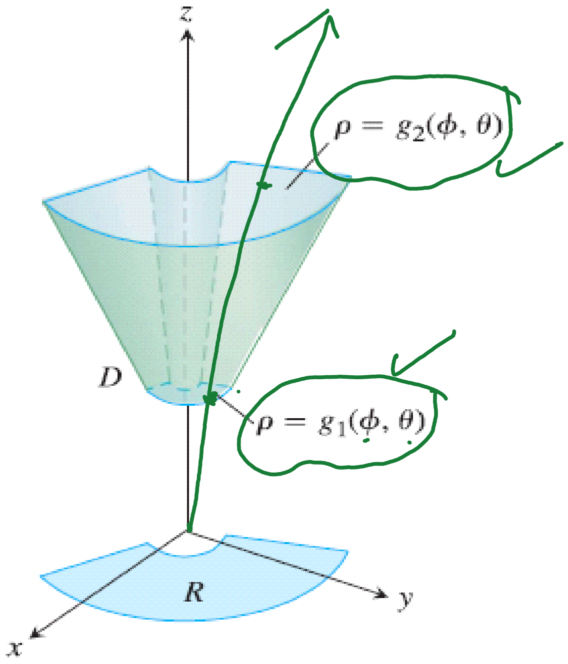
$0 \le \theta \le 2\pi$

How to Integrate in Spherical Coordinates

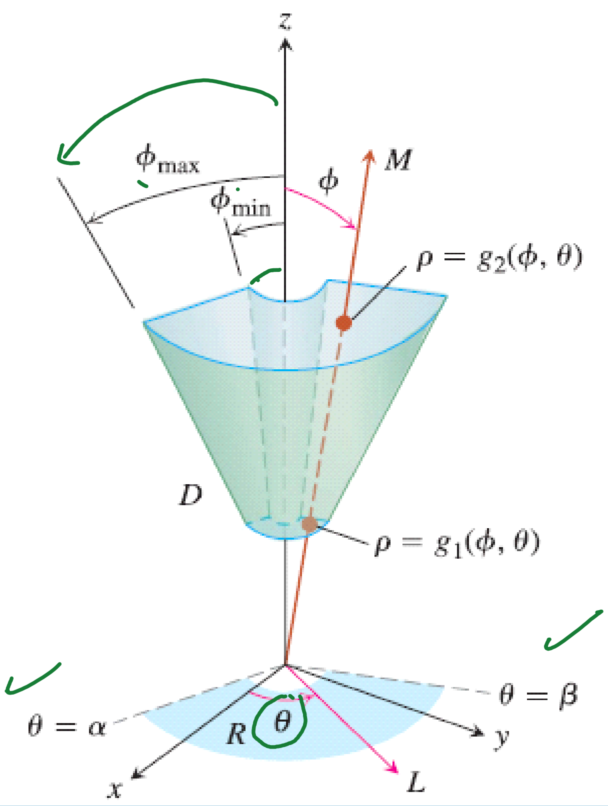
To evaluate

$$\iiint_D f(\rho, \phi, \theta)\, dV$$

1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.



2. *Find the ρ-limits of integration*. Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.

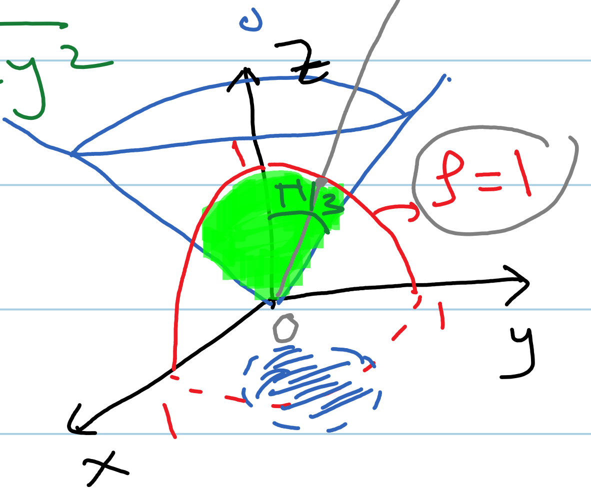


3. *Find the ϕ-limits of integration.* For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are the ϕ -limits of integration.

4. *Find the θ-limits of integration.* The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho,\phi,\theta)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}}\int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} \boxed{f(\rho,\phi,\theta)}\,\underbrace{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}_{dV},$$

Ex. Find the volume of the ice cream cone cut from the solid sphere $\boxed{x^2+y^2+z^2\le 1}$ by the cone $z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$



Sol.

$$x^2+y^2+z^2\le 1$$
$$\rho^2\le 1$$
$$\rho\le 1$$

$$z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2} \Rightarrow \rho\cos\phi=\frac{1}{\sqrt{3}}\sqrt{\rho^2\sin^2\phi}$$

$$\Rightarrow \rho\cos\phi=\frac{1}{\sqrt{3}}\rho\sin\phi,\quad \rho>0,\quad 0\le\phi\le\pi$$

$$\Rightarrow \tan\phi=\sqrt{3}$$

$$\boxed{\phi=\frac{\pi}{3}}$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/3} \frac{\rho^3}{3}\sin\phi\,\Big|_{\rho=0}^{\rho=1}\, d\phi\, d\theta$$

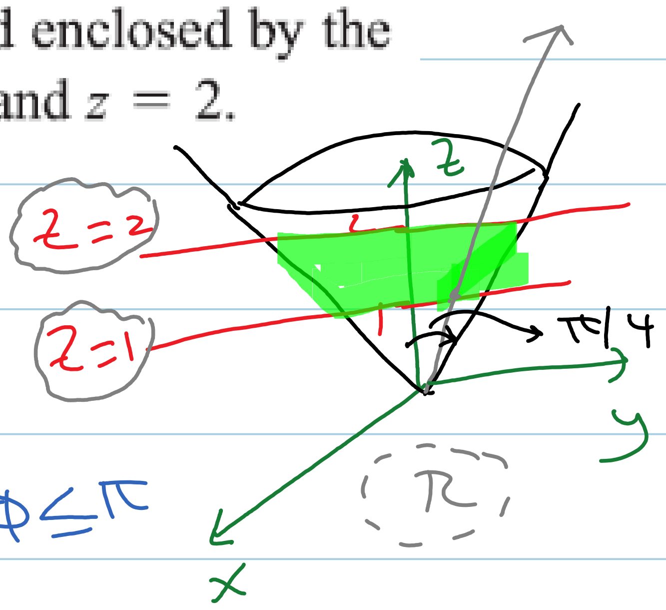
$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}\cos\phi \Big|_{\phi=0}^{\phi=\pi/3} d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\left(\frac{1}{2}\right)+\frac{1}{3}(1)\right] d\theta$$

$$\int_0^{2\pi}\frac{1}{6}\,d\theta = \frac{1}{6}\theta\Big|_0^{2\pi} = \pi/3 .$$

52. Cone and planes Find the volume of the solid enclosed by the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.



Sol. $z = \sqrt{x^2+y^2}$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi}$

$= \rho\sin\phi, \ \rho > 0, \ 0 \le \phi \le \pi$

$\tan\phi = 1$

$\phi = \pi/4$ $\qquad$ $0 \le \phi \le \frac{\pi}{4}.$

$z=1 \Rightarrow \rho\cos\phi = 1 \Rightarrow \rho = \sec\phi$

$z=2 \Rightarrow \rho\cos\phi = 2 \Rightarrow \rho = 2\sec\phi$

$\sec\phi \le \rho \le 2\sec\phi$

$0 \le \theta \le 2\pi.$

Volume

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{2\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

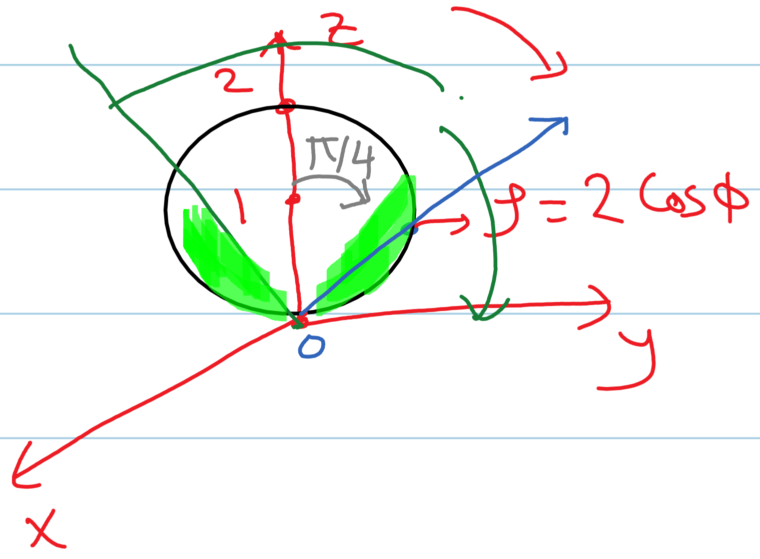
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\phi \Big|_{\rho=\sec\phi}^{\rho=2\sec\phi} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \left[\frac{8}{3}\sec^3\phi\sin\phi - \frac{1}{3}\sin\phi\sec^3\phi\right] d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \left(\frac{8}{3}\frac{\sin\phi}{\cos^3\phi} - \frac{1}{3}\frac{\sin\phi}{\cos^3\phi}\right) d\phi \, d\theta$$

$u = \cos\phi$ ......

Ex. Find the volume of the solid bdd below by the sphere $x^2+y^2+(z-1)^2=1$ and above by the cone $z=\sqrt{x^2+y^2}$



Sol. $\underbrace{x^2+y^2+z^2}-2z=0$

$\rho^2-2\rho\cos\phi=0$

$\boxed{\rho=2\cos\phi}$

$z=\sqrt{x^2+y^2} \Rightarrow \rho\cos\phi=\rho\sin\phi \, , \rho>0$

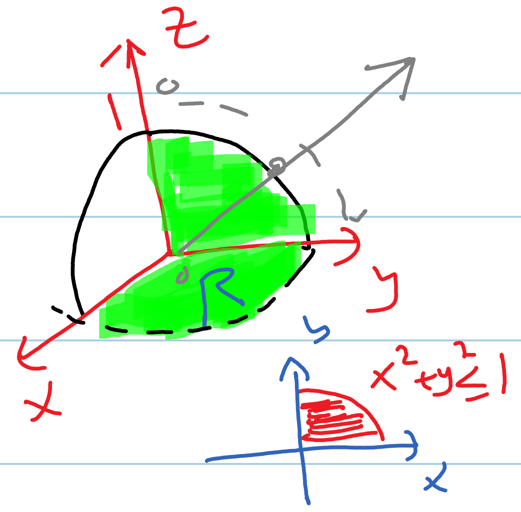
$\tan\phi=1 \Rightarrow \phi=\frac{\pi}{4}$

$0\le\theta\le 2\pi \, , \; \frac{\pi}{4}\le\phi\le\pi/2 \, , \; 0\le\rho\le 2\cos\phi$

$$V=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Ex. Evaluate $$I=\iiint_B x e^{\sqrt{x^2+y^2+z^2}}\,\underbrace{dV}_{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

Where B is the region of the unit ball $x^2+y^2+z^2\le 1$ lies in the first octant.



$x^2+y^2+z^2\le1$

$\boxed{\rho\le1}$

$0\le\rho\le1$

$0\le\phi\le\pi/2$

$0\le\theta\le\pi/2$

$$I=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1\underbrace{\rho\sin\phi\cos\theta\cdot e^{\rho}}_{f}\cdot\underbrace{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}_{dV}$$

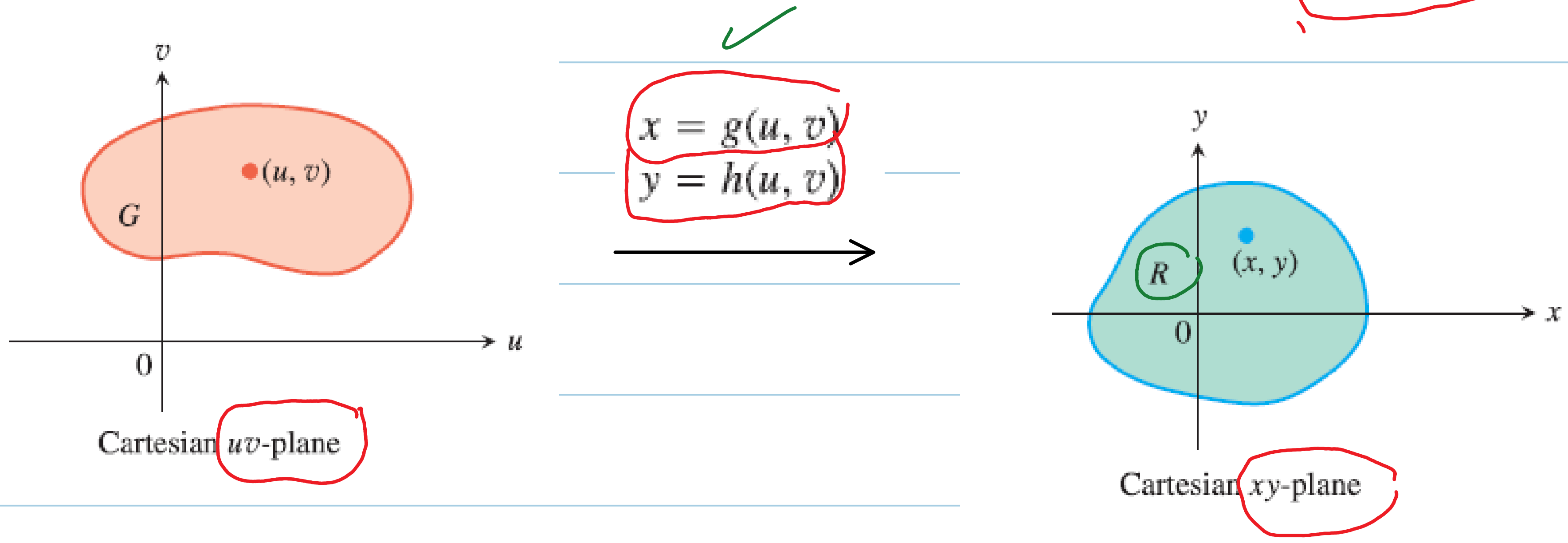
$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1\rho^3e^{\rho}\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$$

$=$ - - - - (H.W)

15.8 Substitutions in Multiple Integrals

$$\int f(g(x)) \cdot g'(x)\,dx \qquad \boxed{u = g(x)}$$

Substitutions in Double Integrals



We call R the **image** of G under the transformation, and G the **preimage** of R.

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(g(u, v), h(u, v))\,\boxed{|J(u, v)|}\,du\,dv. \qquad (1)$$

The factor $J(u, v)$, whose absolute value appears in Equation (1), is the *Jacobian* of the coordinate transformation, named after German mathematician Carl Jacobi.

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \boxed{J(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \qquad (2)$$

The Jacobian can also be denoted by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

EXAMPLE 1 Find the Jacobian for the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, and use Equation (1) to write the Cartesian integral $\iint_R f(x, y)\,dx\,dy$ as a polar integral.

Solution $x = r\cos\theta, \quad y = r\sin\theta$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta)$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r(1) = r.$$

$|J|$

$$\iint_R f(x,y)\,dA = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$\frac{y}{2} \leq x \leq \frac{y}{2}+1$

$0 \leq y \leq 4$

EXAMPLE 2 Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\, dx\, dy$$

by applying the transformation

$$\boxed{u = \frac{2x-y}{2}}, \qquad \boxed{v = \frac{y}{2}}$$

and integrating over an appropriate region in the uv-plane.

Sol.

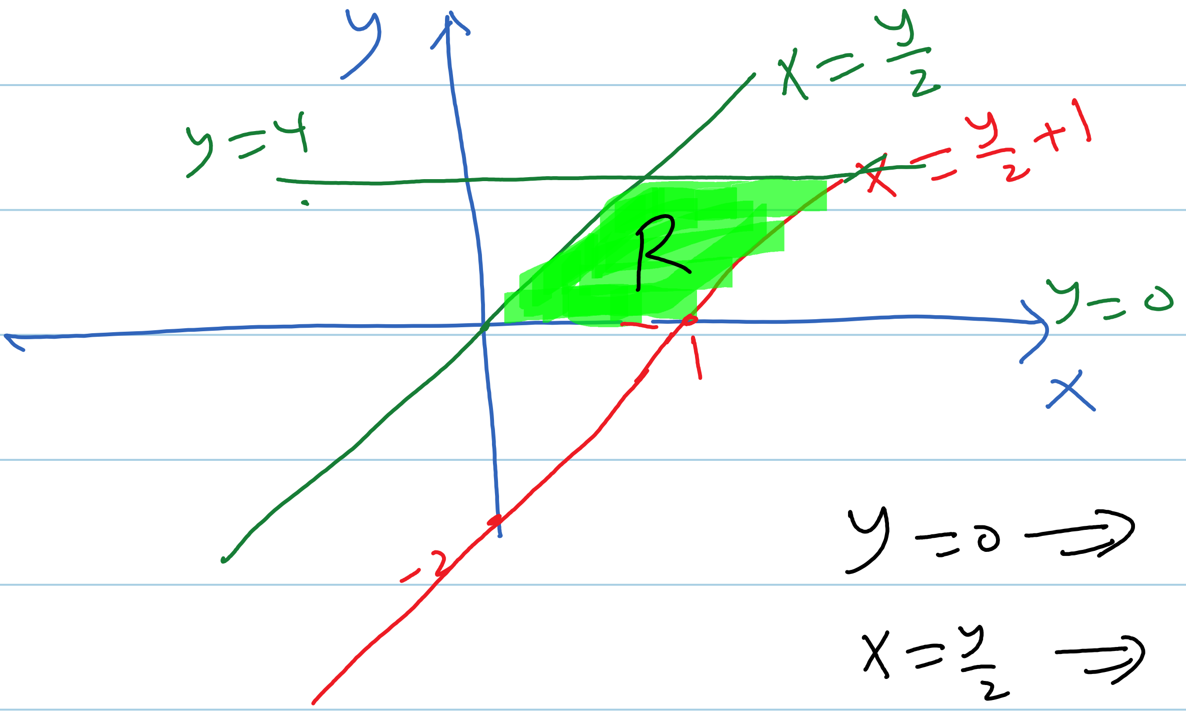
$$\left.\begin{array}{l} 2u = 2x - y \\ 2v = y \end{array}\right\} \Rightarrow 2u + 2v = 2x$$

$$\boxed{x = u + v} \checkmark$$

$$\boxed{y = 2v} \checkmark$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$R: \quad \frac{y}{2} \leq x \leq \frac{y}{2}+1, \quad 0 \leq y \leq 4$



$x = \frac{y}{2}+1$

$y = 0 \Rightarrow x = 1$

$x = 0 \Rightarrow y = -2$

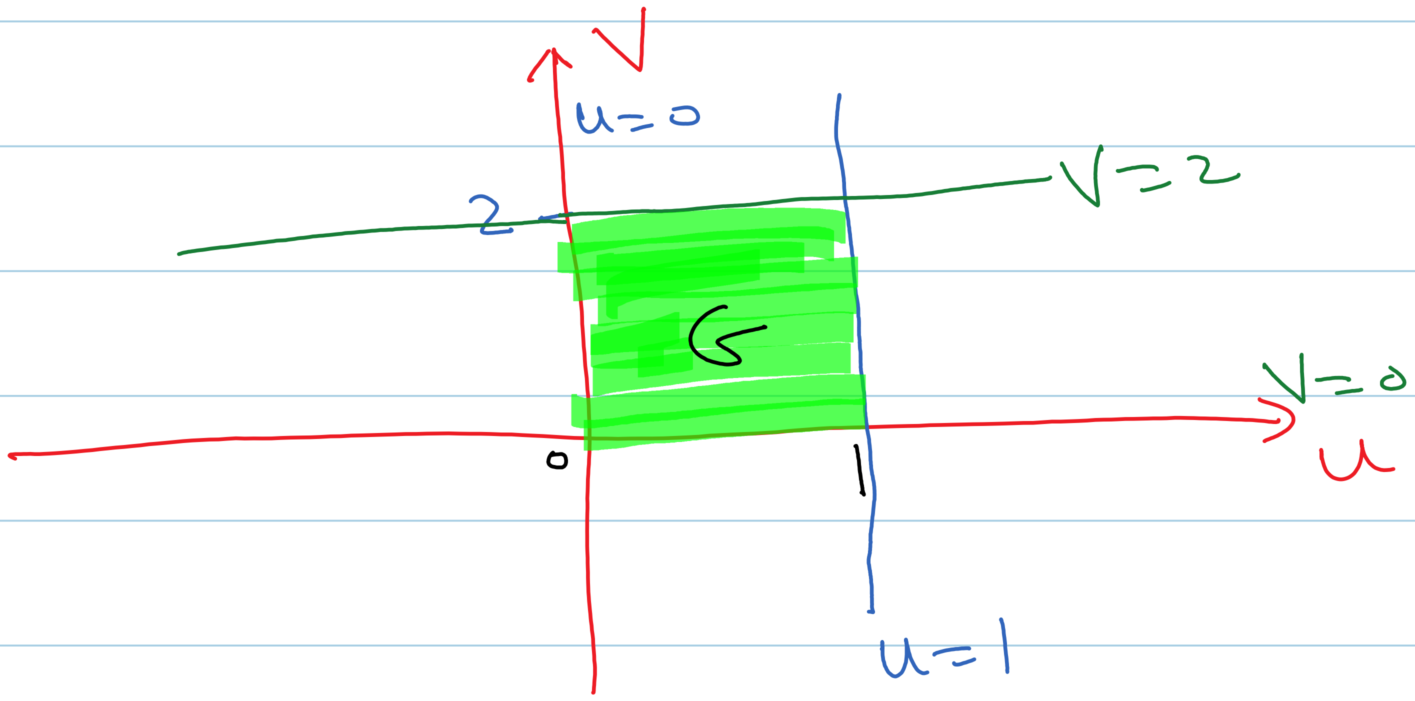
$y = 0 \Rightarrow 2v = 0 \Rightarrow \boxed{v = 0}$

$x = \frac{y}{2} \Rightarrow u + v = \frac{2v}{2} \Rightarrow \boxed{u = 0}$

$x = \frac{y}{2} + 1 \Rightarrow u + v = \frac{2v}{2} + 1$

$\boxed{u = 1}$

$y = 4 \Rightarrow 2V = 4 \Rightarrow \boxed{V = 2}$



$x = u + v$

$y = 2v$

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2} \, dx \, dy = \iint_G \frac{2(u+v) - 2v}{2} |J| \, du \, dv$$

$$= \int_0^2 \int_0^1 u \, |2| \, du \, dv$$

$$= \int_0^2 u^2 \Big|_0^1 dv = \int_0^2 1 \, dv$$

$$= 2.$$

$R: 0 \le y \le 1-x,$
$0 \le x \le 1$

EXAMPLE 3 Evaluate

$$\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dy\,dx.$$

Sol. $u = x+y \quad , \quad V = y-2x$

$$\begin{aligned} x+y &= u \\ -1\,(-2x+y &= V \\ \hline 3x &= u - V \end{aligned} \Rightarrow \boxed{x = \tfrac{1}{3}u - \tfrac{1}{3}V}$$

$$y = u - x = u - \left(\tfrac{1}{3}u - \tfrac{1}{3}V\right)$$

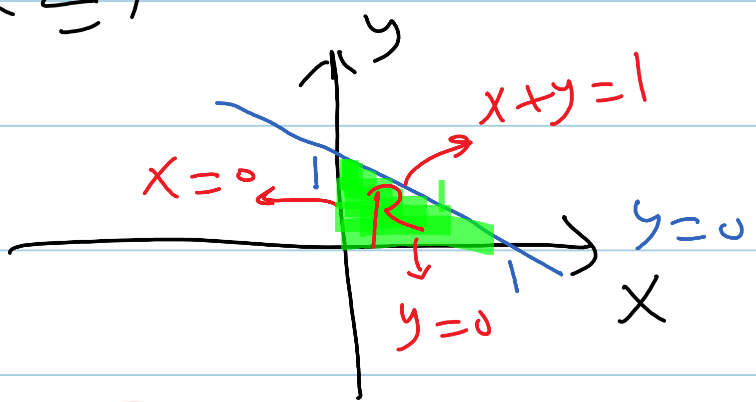
$$\boxed{y = \tfrac{2}{3}u + \tfrac{1}{3}V}$$

$$J(u,v) = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

$R: 0 \le y \le 1-x, \quad 0 \le x \le 1$

$y = 0$

$y = 1-x$

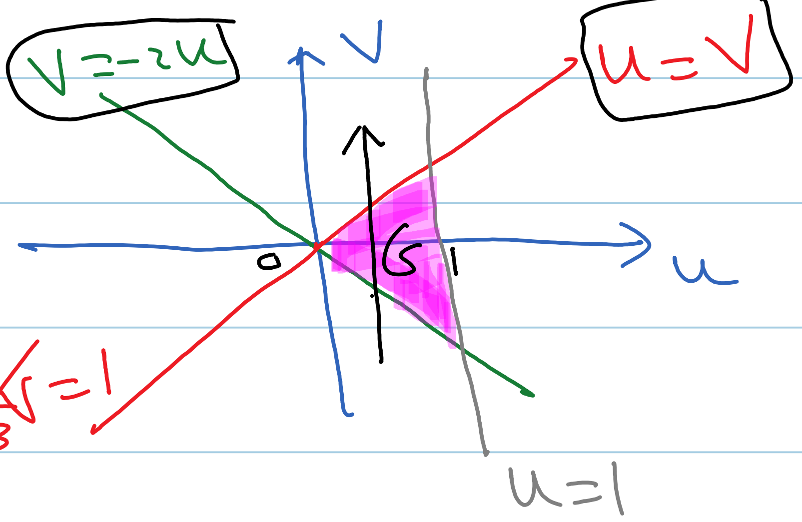


G:

$\boxed{y=0} \Rightarrow \tfrac{2}{3}u + \tfrac{1}{3}V = 0 \Rightarrow \boxed{V = -2u}$

$\boxed{x=0} \Rightarrow \tfrac{1}{3}u - \tfrac{1}{3}v = 0$

$\boxed{u = v}$



$\boxed{x+y=1} \Rightarrow \tfrac{1}{3}u - \cancel{\tfrac{1}{3}V} + \tfrac{2}{3}u + \cancel{\tfrac{1}{3}V} = 1$

$\boxed{u = 1}$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

$$= \int_0^1 \int_{-2u}^{u} \sqrt{u}\;v^2\;|J|\,dv\,du$$

$$= \int_0^1 \underbrace{\int_{-2u}^{u} \sqrt{u}\,v^2 \left|\tfrac{1}{3}\right| dv}\,du$$

$$= \int_0^1 \frac{1}{3}\sqrt{u}\,\frac{v^3}{3}\Big|_{v=-2u}^{v=u}\,du$$

$$= \frac{1}{9}\int_0^1 \left[\sqrt{u}\cdot u^3 - \sqrt{u}\,(-8u^3)\right] du$$

$$= \frac{1}{\cancel{9}} \int_0^1 \cancel{9}\,u^{7/2}\,du$$

$$= \frac{2}{9}\,u^{9/2}\Big|_0^1 = \frac{2}{9}$$

the End of Cal 3

Ch 12, 13, 14, 15

Discussion on Chapter 14

14.1

In Exercises 65–68, find and sketch the domain of f. Then find an equation for the level curve or surface of the function passing through the given point.

65. $f(x,y)=\sum_{n=0}^{\infty}\left(\frac{x}{y}\right)^{n}$, $(1, 2)$

$$\sum_{n=0}^{\infty} x^{n} = 1 + x + x^{2} + \cdots$$

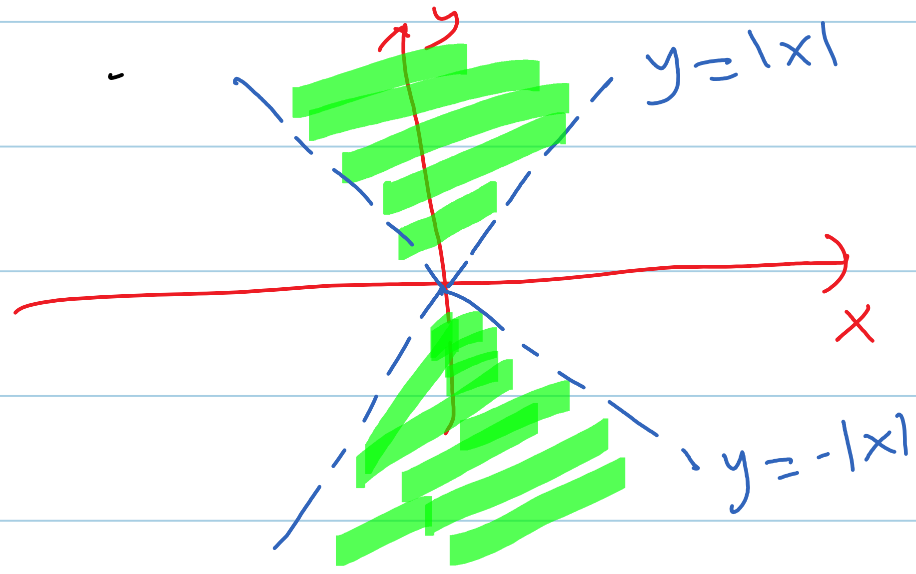
$$= \frac{1}{1-x}, \; |x|<1$$

$$= \frac{1}{1-\frac{x}{y}}, \; \left|\frac{x}{y}\right|<1$$

$$f(x,y) = \frac{y}{y-x}, \; \boxed{|x|<|y|} \qquad \boxed{y \neq x}$$

$$\text{Dom}(f) = \{(x,y) : |y|>|x|\}$$

$y>|x|$ or $y<-|x|$



. $f(x,y)=f(1,2)$

$$\frac{y}{y-x} = \frac{2}{2-1} = 2$$

$$y = 2y - 2x$$

$$\boxed{y = 2x}$$ line

68. $g(x,y,z)=\int_{x}^{y}\frac{dt}{1+t^{2}}+\int_{0}^{z}\frac{d\theta}{\sqrt{4-\theta^{2}}}$, $(0, 1, \sqrt{3})$

$$g(x,y,z) = \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right)$$

$$D_g = \left\{(x,y,z) : -1 \le \frac{z}{2} \le 1\right\}$$

$$= \{(x,y,z) : -2 \le z \le 2\}$$

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Level surface

$g(x,y,z) = g(0,1,\sqrt{3})$

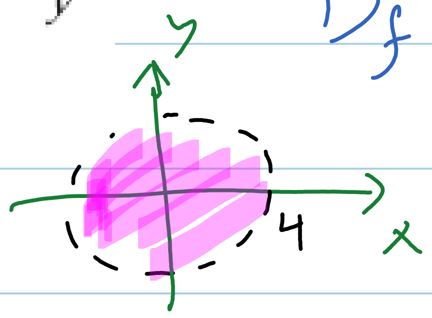
$\tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right) = \frac{\pi}{4} - 0 + \frac{\pi}{3}$

$\tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right) = \frac{7\pi}{12}$

23. $f(x, y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$

$D_f = \{(x,y) : 16 - x^2 - y^2 > 0\}$

$= \{(x,y) : x^2 + y^2 < 16\}$



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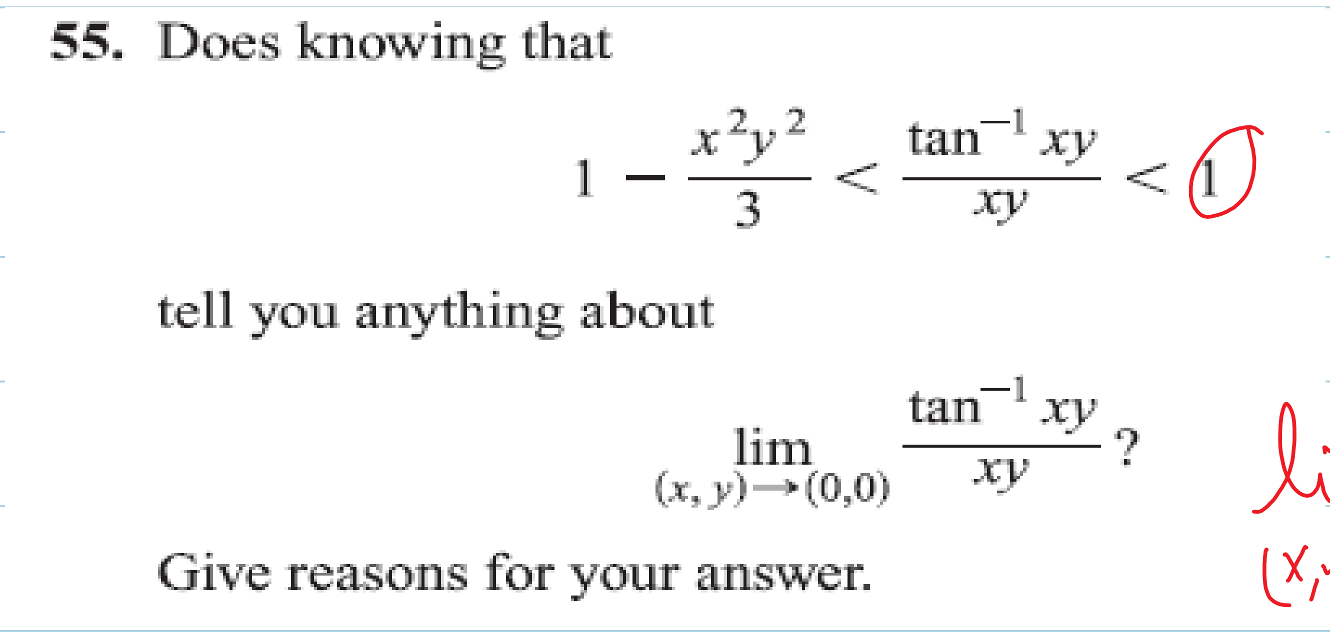
14.2 Limits + Continuity.



55. Does knowing that

$$1-\frac{x^2y^2}{3} < \frac{\tan^{-1}xy}{xy} < 1$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}xy}{xy}\ ?$$

Give reasons for your answer.

$$\lim_{(x,y)\to(0,0)} 1 = 1.$$

$$\lim_{(x,y)\to(0,0)} \left(1-\frac{x^2y^2}{3}\right) = 1$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(xy)}{xy} = 1$$ by Sandwich Thm.

Another method,

$u = xy$, $(x,y)\to(0,0) \Rightarrow u\to 0$

$$\lim_{u\to 0} \frac{\tan^{-1}u}{u} \overset{(\frac{0}{0})}{=} \lim_{u\to 0} \frac{\frac{1}{1+u^2}}{1} = 1.$$

14.3 partial derivatives.

22. $f(x, y) = \sum_{n=0}^{\infty}(xy)^n \quad (|xy| < 1)$

$$= \frac{1}{1-xy}, \ |xy| < 1$$

$$f_x = \frac{-(0-y)}{(1-xy)^2} = \frac{y}{(1-xy)^2}$$

$$f_y = \frac{x}{(1-xy)^2}$$

$z = f(x,y)$ find $\frac{\partial y}{\partial x} = ??$

$$F = f(x,y) - z = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{f_x}{f_y}$$

ex. $z^2 = e^x + y^2$ find $\frac{\partial z}{\partial x}$ ✓

Sol. $F(x,y,z) = e^x + y^2 - z^2 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^x}{-2z}$$

$$= \frac{e^x}{2z}$$

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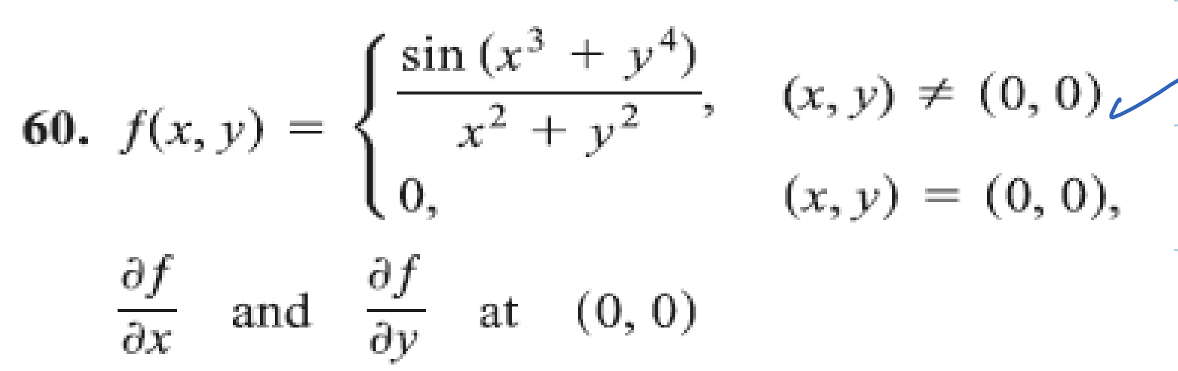

60. $f(x, y) = \begin{cases} \dfrac{\sin(x^3 + y^4)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$ ✓

$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0, 0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - \cancelto{0}{f(0,0)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(h^3)}{h^2}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h^3)}{h^3} = 1.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - \cancelto{0}{f(0,0)}}{k}$$

$$= \lim_{k\to 0} \frac{\sin(k^4)}{k^3}$$

$$= \lim_{k\to 0} \frac{\cos(k^4)\cdot 4k^3}{3k^2}$$

$$= \frac{4}{3}\lim_{k\to 0} k\cos(k^4)$$

$$= \frac{4}{3}(0)(1) = 0.$$

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84. $w = \ln(2x + 2ct)$ Satisfies $\boxed{\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2},}$, c constant

$$\frac{\partial w}{\partial t} = \frac{2c}{2x+2ct} = 2c(2x+2ct)^{-1}$$

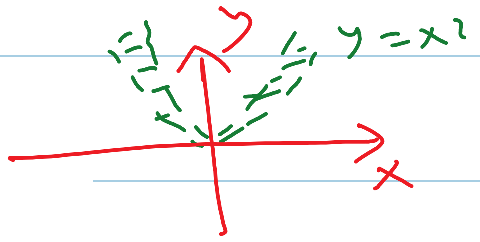
$$\frac{\partial^2 w}{\partial t^2} = -2c(2x+2ct)^{-2} \cdot 2c$$

$$= \frac{-4c^2}{(2x+2ct)^2} = L.H.S$$

$$\frac{\partial w}{\partial x} = \frac{2}{2x+2ct} \Rightarrow \frac{\partial^2 w}{\partial x2} = \frac{-4}{(2x+2ct)^2}$$

$$R.H.S = c^2 \frac{\partial^2 w}{\partial x^2} = \frac{-4c^2}{(2x+2ct)^2} = L.H.S.$$

$$x^2 < x^2 < 2x^2$$



92. Let $f(x, y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise.} \end{cases}$

Show that $\underline{f_x(0, 0) \text{ and } f_y(0, 0)}$ exist, but $\underline{f \text{ is not differentiable at}}$ $(0, 0)$.

<u>Sol.</u> $$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - 1}{h}$$

$$= \lim_{h \to 0} \frac{1-1}{h} = \lim_{h \to 0} \boxed{\frac{0}{h}}$$

$$= \lim_{h \to 0} 0 = 0$$ exists

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 1 = 1$$

Along $y=x^2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 0 = 0$$

Along $y=1.5x^2$

$\Rightarrow$ Two path test $\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ DNE

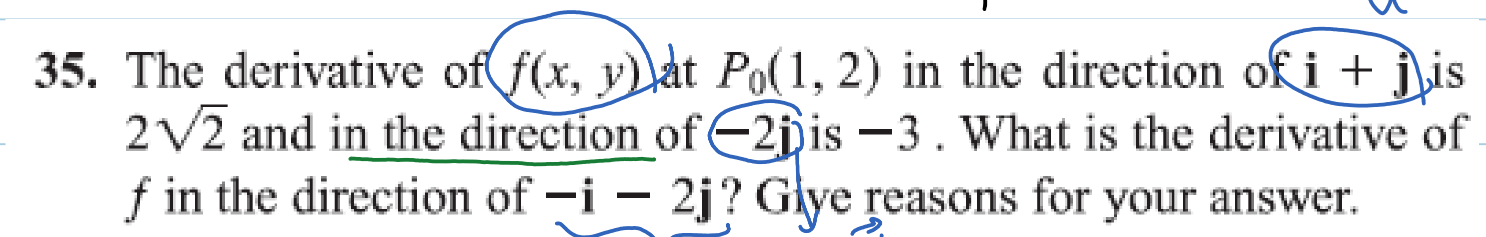
$\Rightarrow$ f is not cont. at (0,0)

$\Rightarrow$ f is not diffble.

14.4 Chain Rule

14.5 Directional Derivatives and Gradient Vectors

$$D_{\vec u} f\Big|_{P_0} = \nabla f\Big|_{P_0} \cdot \vec u \qquad \vec u: \text{unit}$$



35. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\overset{\vec u}{\mathbf{i} + \mathbf{j}}$ is $2\sqrt{2}$ and in the direction of $\underset{\vec v}{-2\mathbf{j}}$ is -3 . What is the derivative of f in the direction of $\underbrace{-\mathbf{i} - 2\mathbf{j}}_{\vec w}$? Give reasons for your answer.

Sol. $D_{\vec u} f\Big|_{(1,2)} = 2\sqrt{2} \Rightarrow \nabla f\Big|_{(1,2)} \cdot \left(\frac{i+j}{\sqrt{2}}\right) = 2\sqrt{2}$

$$(f_x i + f_y j)\cdot(i+j) = 4$$

$$f_x\Big|_{(1,2)} + f_y\Big|_{(1,2)} = 4 \quad \text{----} \ ①$$

$$D_{\vec v} f\Big|_{(1,2)} = -3$$

$$\nabla f\Big|_{(1,2)} \cdot \frac{(-2j)}{2} = -3$$

$$-f_y\Big|_{(1,2)} = -3 \Rightarrow \boxed{f_y\Big|_{(1,2)} = 3}$$

$$\overset{eq(1)}{\Longrightarrow} f_x\Big|_{(1,2)} = 4-3=1$$

$$\therefore \nabla f\Big|_{(1,2)} = i + 3j$$

$$D_{\vec{w}} f\Big|_{(1,2)} = \nabla f\Big|_{(1,2)} \cdot \frac{(-i-2j)}{\sqrt{5}}$$

$$= (i+3j) \cdot \left(\frac{-i-2j}{\sqrt{5}} \right)$$

$$= -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$

14.6 Tangent Planes and Differentials

38. $f(x, y) = \ln x + \ln y$ at $P_0(1, 1)$,

R: $|x - 1| \le 0.2$, $|y - 1| \le 0.2$

Find $L(x,y)$

$|E|$.

Sol. $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$f_x = \frac{1}{x}$, $f_y = \frac{1}{y} \Rightarrow f_x(1,1) = f_y(1,1) = 1$

$f(1,1) = \ln 1 + \ln 1 = 0$

$L(x,y) = 0 + 1(x-1) + 1(y-1) = x+y-2$

$|E| \le \frac{1}{2} M(|x-1| + |y-1|)^2$

$|x-1| < 0.2$

$-0.2 < x-1 < 0.2$

$0.8 < x < 1.2$

$|y-1| < 0.2$

$0.8 < y < 1.2$

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$f_{xx} = -\frac{1}{x^2}$, $f_{yy} = -\frac{1}{y^2}$, $\boxed{f_{xy} = \boxed{0}}$

$|f_{xx}| = \frac{1}{x^2}$ $\boxed{0.8 \le x \le 1.2}$

$\le \frac{1}{(0.8)^2} \approx 1.56$

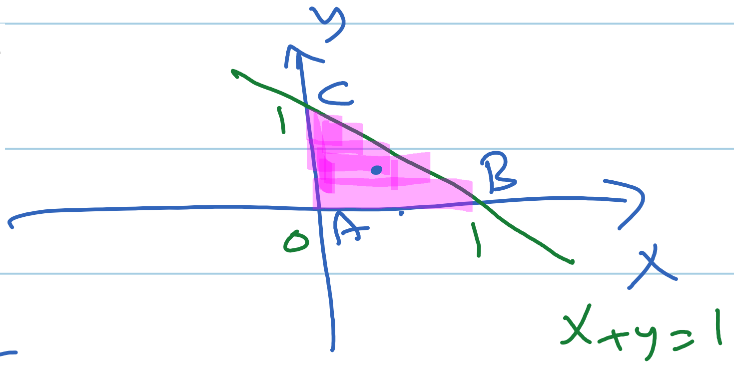
$|f_{yy}| \le \frac{1}{(0.8)^2}$ ✓

$$M = \frac{1}{(0.8)^2} = \frac{100}{64} = \frac{50}{32} = \frac{25}{16} \approx 1.56$$

$$|E| \le \frac{1}{2}\left(\frac{25}{16}\right)(0.2+0.2)^2 \approx \dots -$$

14.7 Extreme Values and Saddle Points

38. $f(x, y) = 4x - 8xy + 2y + 1$ on the triangular plate bounded by the lines $x = 0$, $y = 0$, $x + y = 1$ in the first quadrant



Interior pts.

$f_x = 4 - 8y = 0 \Rightarrow y = \frac{1}{2}$

$f_y = -8x + 2 = 0 \Rightarrow x = \frac{1}{4}$ $\boxed{\left(\frac{1}{4}, \frac{1}{2}\right)}$ ✓

Boundaries $\overline{AB}$ $\boxed{y = 0}$, $0 \le x \le 1$

$f(x,0) = 4x + 1 \Rightarrow f' = 4 \ne 0$

Endpts $\boxed{(0,0)}$, $\boxed{(1,0)}$

$\overline{AC}$: $x = 0$, $0 \le y \le 1$

$f(0,y) = 2y + 1 \Rightarrow f' = 2 \ne 0$

Endpts $(0,0)$, $\boxed{(0,1)}$ ✓

$\overline{BC}$ $x+y=1 \Rightarrow y=1-x \, , \, 0\le x\le 1$

$f(x,y)=f(x,1-x)=4x-8x(1-x)+2(1-x)+1$

$=4x-8x+8x^2+2-2x+1$

$=8x^2-6x+3$

$f'=16x-6=0 \Rightarrow \boxed{x=\frac{3}{8}}$ ✓

$y=1-\frac{3}{8}=\frac{5}{8}$ ✓

$\left(\frac{3}{8},\frac{5}{8}\right)$

Endpoints $(0,1)$, $(1,0)$

مكرر مكرر

عوّض بالنقاط في f أكبر max ، أصغر min

14.8 Lagrange Multipliers

25. Minimizing a sum of squares Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

Constraints

Sol. $x+y+z=9 \Rightarrow g(x,y,z) = x+y+z-9$

Let $f(x,y,z) = x^2+y^2+z^2$

$$\nabla f = \lambda \nabla g$$

$$2xi+2yj+2zk = \lambda(i+j+k)$$

$2x=\lambda$ —— ① $\Rightarrow x=\frac{\lambda}{2}$

$2y=\lambda$ —— ② $y=\frac{\lambda}{2}$

$2z=\lambda$ —— ③ $z=\frac{\lambda}{2}$

$x+y+z=9$ —— ④

①, ②, ③ into ④:

$$\frac{\lambda}{2}+\frac{\lambda}{2}+\frac{\lambda}{2}=9 \Rightarrow \frac{3\lambda}{2}=9$$

$$\boxed{\lambda=6}$$

$$x=y=z=\frac{6}{2}=3.$$

the min. value of f is

$$f(3,3,3)=3^2+3^2+3^2=27.$$